A Topological Approach to Cheeger-Gromov Universal Bounds for von Neumann ρ-Invariants

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POSTECH

Abstract

Using deep analytic methods, Cheeger and Gromov showed that for any smooth $(4k-1)$-manifold there is a universal bound for the von Neumann L^2 ρ-invariants associated to arbitrary regular covers. We present a proof of the existence of a universal bound for topological $(4k-1)$-manifolds, using L^2-signatures of bounding $4k$-manifolds. We give explicit linear universal bounds for 3-manifolds in terms of triangulations, Heegaard splittings, and surgery descriptions. We show that our explicit bounds are asymptotically optimal. As an application, we give new lower bounds of the complexity of 3-manifolds that can be arbitrarily larger than previously known lower bounds. As ingredients of the proofs that seem interesting on their own, we develop a geometric construction of efficient 4-dimensional bordisms of 3-manifolds over a group and develop an algebraic topological notion of uniformly controlled chain homotopies. © 2015 Wiley Periodicals, Inc.

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1 Introduction and Main Results

In [13], Cheeger and Gromov studied the L^2 ρ-invariant $\rho^{(2)}(M, \phi) \in \mathbb{R}$, which they defined for a closed $(4k-1)$-dimensional smooth manifold M and a homomorphism $\phi : \pi_1(M) \to G$ to a group G. Briefly speaking, for a Riemannian

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metric on M, $\rho^{(2)}(M,\phi)$ is the difference of the η-invariant of the signature operator of M and the L^2 η-invariant of that of the G-cover of M which is defined using the von Neumann trace. As a key ingredient of their study of topological invariance, Cheeger and Gromov showed that there is a universal bound of the L^2 η-invariants of arbitrary coverings of M by using deep *analytic* methods. Equivalently, there is a universal bound on the Cheeger-Gromov ρ-invariants of M:

THEOREM 1.1 (Cheeger-Gromov [13]). *For any closed, smooth* $(4k-1)$*-manifold* M*, there is a constant* C_M *such that* $|\rho^{(2)}(M,\phi)| \leq C_M$ *for any homomorphism* $\phi : \pi_1(M) \to G$ *to any group* G*.*

In this paper we develop a *topological* approach to the Cheeger-Gromov universal bound C_M. Our method presents a topological proof of the existence and gives new topological understanding of the universal bound with applications to low-dimensional topology. In particular, we reveal a relationship of the Cheeger-Gromov ρ-invariant and the complexity theory of 3-manifolds.

In this section, we discuss some background and motivations, state our main results and applications, and introduce some ingredients of the proofs developed in this paper that seem interesting on their own. In particular, we introduce an algebraic topological notion of controlled chain homotopy in Section 1.5.

As a convention, we assume that manifolds are compact and oriented unless stated otherwise.

1.1 Background and Motivation

A known approach to ρ-invariants is to use a standard index theoretic fact that if a $(4k-1)$-manifold M is the boundary of a $4k$-manifold W to which the given representation of $\pi_1(M)$ extends, then the ρ-invariant of M may be computed as a signature defect of W. For the von Neumann L^2 case, as first appeared in the work of Chang and Weinberger [12], we can recast this index theoretic *computation* to provide a topological *definition*: for any M and ϕ, $\rho^{(2)}(M,\phi)$ can be defined as a topological L^2-signature defect of a certain bounding manifold, in the topological category as well as the smooth category. This is done using a theorem of Kan and Thurston that an arbitrary group embeds into an acyclic group [31] and using the invariance of the von Neumann trace under composition with a monomorphism. Also, instead of Hilbert modules and L^2-(co)homology, we can use standard homology over the group von Neumann algebra by employing the L^2-dimension theory of Lück [35, 36]. For the reader's convenience, we provide precise definitions and detailed arguments in Section 2.1 for topological $(4k-1)$-manifolds.

Although the Cheeger-Gromov ρ-invariant can be defined topologically, known proofs of the existence of a universal bound are entirely analytic [13, 46] and provide hardly any information on the topology of M. From this a natural question arises:

QUESTION 1.2. Can we understand the Cheeger-Gromov bound topologically?

This question is intriguing on its own, along the long tradition of the interplay between geometry and topology. Attempts to understand the Cheeger-Gromov bound using L^2-signature defects have failed (for instance, see [19, p. 348]). The key reason is that the bounding $4k$-manifold used to define $\rho^{(2)}(M, \phi)$ in known arguments depends on the choice of ϕ.

Topological understanding of the Cheeger-Gromov bound is also of importance for applications, particularly to knots, links, and low-dimensional manifolds. Since the work of Cochran, Orr, and Teichner on knot concordance [17], several recently discovered rich structures on topological concordance of knots and links, topological homology cobordism of 3-manifolds, and symmetric Whitney towers and gropes in 4-manifolds have been understood by using the Cheeger-Gromov invariant. The most general known obstructions from the Cheeger-Gromov invariant in this context are given as the amenable signature theorems in [10, theorems 1.1 and 7.1] and [5, theorem 3.2]. In many applications, it is essential to control $\rho^{(2)}(M, \phi)$ for certain homomorphisms ϕ. In [19], Cochran and Teichner first introduced the influential idea that the Cheeger-Gromov bound is useful for this purpose. Since then, the Cheeger-Gromov bound has been used as a key ingredient in various interesting works (some of them are discussed in Remark 6.6). It is known that many existence theorems in these works could be improved to give explicit examples if we had a better understanding of the Cheeger-Gromov bound. A key question arising in this context is the following: if M is the zero surgery manifold of a given knot K, how large is C_M? For instance, for the simplest ribbon knot $K = 6_1$ (the stevedore knot), is C_M less than a billion?

In spite of these desires, almost nothing beyond its existence was known about the Cheeger-Gromov bound.

1.2 Main Results on the Cheeger-Gromov Universal Bound

As our first result, we present a topological proof of the existence of the Cheeger-Gromov bound that directly applies to topological manifolds, based on the L^2-signature defect approach.

THEOREM 1.3. *For any closed topological* $(4k-1)$*-manifold* M*, there is a constant* C_M *such that* $|\rho^{(2)}(M, \phi)| \leq C_M$ *for any homomorphism* $\phi : \pi_1(M) \to G$ *to any group* G*.*

The outline of the proof is as follows. As the heart of the argument, we show that for an arbitrary $(4k-1)$-manifold M, there is a *single* $4k$-manifold W with $\partial W = M$ from which *every* Cheeger-Gromov invariant $\rho^{(2)}(M, \phi)$ of M can be computed as an L^2-signature defect. Once it is proven, it follows that twice the number of 2-cells in a CW structure of W is a Cheeger-Gromov bound, by using the observation that any L^2-signature of W is not greater than the number of 2-cells. A key ingredient used to show the existence of W is a *functorial* embedding of groups into acyclic groups due to Baumslag, Dyer, and Heller [3]. More details are discussed in Section 2.

Beyond giving a topological proof of the existence, our approach provides us a new topological understanding of the Cheeger-Gromov bound. For 3-manifolds, we relate the Cheeger-Gromov bound to the fundamental 3-manifold presentations—*triangulations*, *Heegaard splittings*, and *surgery on framed links*—by giving explicit estimates in terms of topological complexities defined from combinatorial, group theoretic, and knot theoretic information, respectively.

Regarding triangulations, we consider the following natural combinatorial measure of how complicated a 3-manifold is topologically. In this paper, a triangulation designates a simplicial complex structure.

DEFINITION 1.4. The *simplicial complexity* of a 3-manifold M is the minimal number of 3-simplices in a triangulation of M.

The following result relates the combinatorial data to the Cheeger-Gromov bound, which was analytic, via a topological method.

THEOREM 1.5. *Suppose M is a closed 3-manifold with simplicial complexity n. Then*

$$|\rho^{(2)}(M,\phi)| \le 363\,090 \cdot n$$

for any homomorphism $\phi : \pi_1(M) \to G$ to any group G.

In the next subsection, we will discuss an application of Theorem 1.5 to the complexity theory of 3-manifolds. In the last two subsections of this introduction, we will introduce two key ingredients of the proof of Theorem 1.5 (and Theorems 1.8 and 1.9 below), which are essentially topological and algebraic, respectively.

The linear bound given in Theorem 1.5 is *asymptotically optimal*. To state it formally, we define the "most efficient" Cheeger-Gromov bound as a function $B^{\mathrm{sc}}(n)$ in the simplicial complexity n, as follows:

$$B^{\mathrm{sc}}(n) = \sup\left\{ |\rho^{(2)}(M,\phi)| \;\middle|\; \begin{array}{l} M \text{ has simplicial complexity} \le n \\ \text{and } \phi \text{ is a homomorphism of } \pi_1(M) \end{array} \right\}.$$

Theorem 1.5 tells us that $B^{\mathrm{sc}}(n)$ is at most linear asymptotically. In other words, $B^{\mathrm{sc}}(n) \in O(n)$; recall that $f(n) \in O(g(n))$ if $\limsup_{n\to\infty} |f(n)/g(n)| < \infty$. In our case, by Theorem 1.5, we have

$$\limsup_{n\to\infty} \frac{B^{\mathrm{sc}}(n)}{n} \le 363\,090.$$

Also, recall that the small o notation formalizes the notion that $f(n)$ is strictly smaller than $g(n)$ asymptotically; that is, $f(n)$ is dominated by $g(n)$: we say $f(n) \in o(g(n))$ if $\lim_{n\to\infty} |f(n)/g(n)| = 0$. As another standard notation, we say that $f(n) \in \Omega(g(n))$ if $f(n)$ is not dominated by $g(n)$, that is,

$$\limsup_{n\to\infty} |f(n)/g(n)| > 0.$$

We prove the following result in Section 7.2.

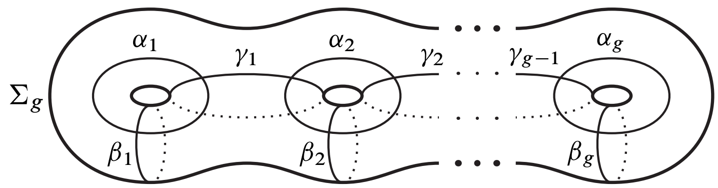


FIGURE 1.1. Lickorish's Dehn twist curves.

THEOREM 1.6. *$B^{\mathrm{sc}}(n) \in \Omega(n)$. In fact,*

$$\limsup_{n\to\infty} \frac{B^{\mathrm{sc}}(n)}{n} \geq \frac{1}{288}.$$

Recall that a Heegaard splitting of a closed 3-manifold is determined by a mapping class h in the mapping class group $\mathrm{Mod}(\Sigma_g)$ of a surface Σ_g of genus g. To make it precise, we use the following convention. We fix a standard embedding of Σ_g into ,S^3 as in Figure 1.1. Let H_1, H_2 be the inner and outer handlebody that Σ_g bounds in $\mathbb{S}^3$, let $i_j : \Sigma_g \to H_j$ $(j = 1, 2)$ be the inclusion, and let α_i and β_i be the basis curves in Figure 1.1. Then the mapping class $h \in \mathrm{Mod}(\Sigma_g)$ of a homeomorphism $f : \Sigma_g \to \Sigma_g$ gives a Heegaard splitting $(\Sigma_g, \{\beta_i\}, \{f(\alpha_i)\})$ of the 3-manifold

$$M = (H_1 \cup H_2)/i_1(f(x)) \sim i_2(x),\ x \in \Sigma_g.$$

In other words, M is obtained by attaching g 2-handles to H_1 along the curves $f(\alpha_i)$ and then attaching a 3-handle. Note that the identity mapping class gives us $\mathbb{S}^3$.

A natural way to measure its complexity is to consider the word length of h in the group $\mathrm{Mod}(\Sigma_g)$. It is well known that $\mathrm{Mod}(\Sigma_g)$ is finitely generated by standard Dehn twists; Lickorish showed that $\mathrm{Mod}(\Sigma_g)$ is generated by the ± 1 Dehn twists about the $3g - 1$ curves α_i, β_i, and γ_i shown in Figure 1.1 [34].

DEFINITION 1.7. The *Heegaard-Lickorish complexity* of a closed 3-manifold M is defined to be the minimal word length, with respect to the Lickorish generators, of a mapping class $h \in \mathrm{Mod}(\Sigma_g)$ that gives a Heegaard splitting of M.

The above geometric group theoretic data is related to the Cheeger-Gromov bound by the following result, which we obtain by combining Theorem 1.5 with a result in [6] (see Section 6.2).

THEOREM 1.8. *If M is a closed 3-manifold with Heegaard-Lickorish complexity ℓ, then*

$$|\rho^{(2)}(M, \phi)| \leq 251\,258\,280 \cdot \ell$$

for any homomorphism $\phi : \pi_1(M) \to G$ to any group G.

We also relate the Cheeger-Gromov bound to surgery presentations of 3-manifolds given as framed links. For a framed link L in $\mathbb{S}^3$, let $n_i(L) \in \mathbb{Z}$ be the framing on the i^{th} component L_i, that is, $n_i(L) = \operatorname{lk}(L_i, L_i')$ where L_i' is the parallel copy of L_i taken along the given framing. We define $f(L) = \sum_i |n_i(L)|$. We denote by $c(L)$ the *crossing number* of a link L in $\mathbb{S}^3$, that is, the minimal number of crossings of a planar diagram of L.

THEOREM 1.9. *Suppose M is a 3-manifold obtained by surgery along a framed link L in $\mathbb{S}^3$. Then*

$$|\rho^{(2)}(M,\phi)| \le 69\,713\,280 \cdot c(L) + 34\,856\,640 \cdot f(L)$$

for any ,homomorphism $\phi : \pi_1(M) \to G$ to any group G.

The proof is given in Section 6.2.

Similarly to Theorem 1.6, we show that the linear bounds in Theorems 1.8 and 1.9 are asymptotically optimal. For formal statements and proofs, see Definition 7.3, Theorem 7.4, and related discussions in Section 7.1.

Remark 1.10. While the linear bounds in Theorems 1.5, 1.8, and 1.9 are asymptotically optimal, it seems that the *coefficients* in these linear bounds can be improved. Although we do not address it in this paper, finding optimal or improved coefficients seems to be an interesting problem.

As an application, our explicit universal bounds for the Cheeger-Gromov invariants are useful in improving several recent results in low-dimensional topology related to knots, links, 3-manifolds, and their 4-dimensional equivalence relations. For instance, by our results above, the proofs of numerous existence results in [4, 5, 8, 11, 14–16, 19, 22, 32] now give explicit examples. See Remark 6.6 for more details.

1.3 Applications to Lower Bounds of the Complexity of 3-Manifolds

The notion of the complexity of 3-manifolds has been an intriguing subject of study. In the literature, the following variation of the simplicial complexity is often considered: a *pseudosimplicial triangulation* of a 3-manifold is defined to be a collection of 3-simplices whose faces are identified in pairs under affine homeomorphisms to give the 3-manifold as a quotient space. Similarly to Definition 1.4, the *pseudosimplicial complexity* $c(M)$ of a 3-manifold M is defined to be the minimal number of 3-simplices in a pseudosimplicial triangulation. Following conventions in the literature, we call $c(M)$ the *complexity* of M (we use the terminology *simplicial complexity* in Definition 1.4 to avoid confusion.) In [39], Matveev defines the notion of complexity using spines in 3-manifolds, which turns out to be equal to $c(M)$ for closed irreducible 3-manifolds M except $M = \mathbb{S}^3, \mathbb{R}P^3$, and $L(3,1)$, and develops some fundamental results.

Finding an efficient (pseudosimplicial) triangulation is essential to several aspects of 3-manifold topology, from the normal surface theory initiated in the 1920s by Kneser to recent quantum invariants and computational approaches. Nonetheless, understanding the complexity for the general case remains as a difficult problem. While we easily obtain an upper bound from a triangulation, finding a lower bound has been recognized as a hard problem [30, 40].

We briefly overview known results on lower bounds of $c(M)$. In [41], Matveev and Pervova obtain basic lower bounds of $c(M)$ from $H_1(M)$ and from the presentation length of $\pi_1(M)$ (see the end of Section 7.1). We remark that in most cases finding the presentation length of a group is another hard problem. In [42], Matveev, Petronio, and Vesnin observe and use that for a hyperbolic 3-manifold M, the Gromov norm $\mathrm{vol}(M)/v_3$ is a lower bound for $c(M)$, where v_3 is the volume of a regular ideal tetrahedron in $\mathbb{H}^3$. In a series of papers [28–30], Jaco, Rubinstein, and Tillmannn develop remarkable techniques to understand the complexity, particularly to find lower bounds, using double covers and a $\mathbb{Z}_2$-version of the Thurston norm.

As an application of our results on the Cheeger-Gromov bound, we present new lower bounds of the complexity of 3-manifolds. For the simplicial complexity, note that Theorem 1.5 already told us that for any homomorphism ϕ of $\pi_1(M)$

$$\frac{1}{363\,090} \cdot |\rho^{(2)}(M,\phi)|$$

is a lower bound. Since the second barycentric subdivision of a pseudosimplicial triangulation is a simplicial complex and since each tetrahedron in a pseudosimplicial triangulation gives $(4!)^2 = 576$ tetrahedra in its second barycentric subdivision, we immediately obtain the following corollary of Theorem 1.5:

COROLLARY 1.11. *If M is a closed 3-manifold, then for any homomorphism ϕ of $\pi_1(M)$,*

$$c(M) \geq \frac{1}{209\,139\,840} \cdot |\rho^{(2)}(M,\phi)|.$$

Although the constant factor in the above inequality is small, the Cheeger-Gromov ρ-invariants of 3-manifolds are often so large that they give interesting new results. First, we have the following:

THEOREM 1.12. *There are 3-manifolds M for which the lower bound for $c(M)$ in Corollary 1.11 is arbitrarily larger than the lower bound information from* (i) *the fundamental group and first homology* [41], (ii) *the hyperbolic volume* [42], *and* (iii) *double covers and $\mathbb{Z}_2$ Thurston norm* [28–30].

In fact, there are 3-manifolds for which the lower bound in Corollary 1.11 grows linearly while the lower bounds in [28–30, 41, 42] vanish or have logarithmic or square root growth. More details are discussed in Section 7.

As an infinite family of explicit examples, we consider lens spaces. In [28, 29], Jaco, Rubinstein, and Tillmann determine the complexity of $L(p,q)$ in certain cases for which p is even, including the case of $L(2k,1)$. Nonetheless, for the general case, current understanding of the complexity of lens spaces is far from complete. In particular, for $L(n,1)$ with n odd, it turns out that previously known lower bounds are not sharp even asymptotically. (For more details, see the discussion at the end of Section 7.1.) In [27, 39], it was conjectured that for $p > q > 0$, $p > 3$, if we write p/q as a continued fraction $[n_0, n_1, \dots]$, then the complexity $c(L(p,q))$ is equal to $\sum n_i - 3$. It specializes to the following:

CONJECTURE 1.13 ([27, 39]). *For $n > 3$, $c(L(n,1)) = n-3$.*

In [27], Jaco and Rubinstein show that $c(L(n,1)) \le n-3$. In [28], Jaco, Rubinstein, and Tillmann prove Conjecture 1.13 for even n. The case of odd n is still open.

In the following result, we give a new lower bound for $c(L(n,1))$ for odd n, which tells us that $c(L(n,1))$ with an arbitrary n is asympotically linear. Recall that we say $f(n) \in \Theta(g(n))$ if the asymptotic growth of $f(n)$ and $g(n)$ are identical; that is, there exist C_1, $C_2 > 0$ such that $C_1|g(n)| \le |f(n)| \le C_2|g(n)|$ for all sufficiently large n.

THEOREM 1.14. $c(L(n,1)) \in \Theta(n)$. *In fact, for each $n > 3$,*

$$\frac{1}{627\,419\,520} \cdot (n-3) \le c(L(n,1)) \le n-3.$$

Theorem 1.14 supports Conjecture 1.13 by telling us that it is asymptotically true.

The proof of Theorem 1.14 employs the Cheeger-Gromov invariants using Corollary 1.11. More applications of our results to the complexity of 3-manifolds will appear in subsequent papers. For instance, in [7], we determine the asymptotic growth of the complexity of surgery manifolds of knots.

1.4 Efficient 4-Dimensional Bordisms over a Group

One of the key ingredients of the proofs of Theorems 1.5, 1.8, and 1.9 is a new result on the existence of an efficient 4-dimensional bordism over a group. More precisely, we address the following problem, which looks interesting on its own.

We consider manifolds over a group G, namely manifolds endowed with a map to BG, the classifying space of G. As usual, we say that W is a *bordism over G between M and N* if $\partial W = M \sqcup -N$ as manifolds over G.

QUESTION 1.15. Given a 3-manifold M over G, how efficiently can M be bordant to a 3-manifold that is over G via a constant map?

To define the efficiency of a bordism rigorously, we consider the following natural notion of complexity of a (co)bordism, which is useful for the study of signature invariants.

DEFINITION 1.16. The 2-*handle complexity* of a 4-dimensional smooth/PL (co)-bordism is the minimal number of 2-handles in a handle decomposition of W.

Although Definition 1.16 (as well as Question 1.15) generalizes to higher dimensions in an obvious way, in this paper we focus on the low-dimensional case only.

It is a standard fact that any L^2-signature of a 4-manifold (in particular the ordinary signature) is not greater than the 2-handle complexity.

Suppose M is a triangulated 3-manifold endowed with a cellular map $\phi : M \to BG$, and $\zeta_M \in C_3(M)$ is the sum of the oriented 3-simplices representing the fundamental class. Then the Atiyah-Hirzebruch bordism spectral sequence tells us that the existence of a bordism W from M to another 3-manifold that is over G via a constant map is equivalent to the existence of a chain level analogue: such W exists if and only if there exists a 4-chain $u \in C_4(BG)$ satisfying $\partial u = \phi_\#(\zeta_M)$. For the reader's convenience we discuss details as Lemma 3.2 in Section 3.1.

Our result (Theorem 3.9 stated below) concerning Question 1.15 is essentially that if the chain level analogue $u \in C_4(BG)$ of a desired W exists for (M, ϕ), then there exists a corresponding bordism W whose 2-handle complexity is controlled *linearly* in the "size" of u and M. To measure the size of a chain, we define an algebraic notion of diameter as follows:

DEFINITION 1.17. Suppose C_* is a based chain complex over $\mathbb{Z}$, and $\{e_\alpha^k\}$ is the given basis of C_k. The *diameter* $d(u)$ of a k-chain $u = \sum_\alpha n_\alpha e_\alpha^k \in C_k$ is defined to be the L^1-norm $d(u) = \sum_\alpha |n_\alpha|$.

Note that the number of tetrahedra in a triangulation of a closed 3-manifold M is equal to the diameter of the chain $\zeta_M \in C_3(M)$ representing the fundamental class.

In order to use the notion of the diameter for a chain in BG (particularly in Theorem 3.9 stated below), we need to fix a CW structure of BG. It is known that we can obtain a $K(G, 1)$ space BG as the geometric realization of the simplicial classifying space of G (i.e., the nerve) that is a simplicial set. Due to Milnor [44], this gives us an explicit CW structure for BG. In addition, Milnor's geometric realization tells us that each n-cell of BG is naturally identified with the standard n-simplex. Another useful fact is that any map of a simplicial complex to BG is homotopic to a cellular map that, roughly speaking, sends simplices to simplices affinely; we call such a map *simplicial-cellular*. We give precise definitions and provide more details in Section 3.2 and in the appendix (in particular, see Definition 3.6).

Now we can state our result about Question 1.15.

THEOREM (A Special Case of Theorem 3.9). *Suppose M is a triangulated closed 3-manifold with $d(\zeta_M)$ tetrahedra, and M is over G via a simplicial-cellular map $\phi : M \to BG$. If there is a 4-chain $u \in C_4(BG)$ satisfying $\partial u = \phi_\#(\zeta_M)$, then there exists a smooth bordism W, between M and a 3-manifold, which is over G via a constant map whose 2-handle complexity is at most* $195 \cdot d(\zeta_M) + 975 \cdot d(u)$.

Our proof provides a geometric construction of a desired bordism W using transversality and surgery arguments over G. It may be viewed as a "geometric realization" of the algebraic idea of the Atiyah-Hirzebruch bordism spectral sequence constructed from the exact couple arising from skeleta. To control the 2-handle complexity of W carefully, we carry out transversality and surgery arguments *simplicially*. Details can be found in Section 3.

We also show that the linear 2-handle complexity in (the special case of) Theorem 3.9 is asymptotically the best possible. For precise statements and detailed discussions, see Section 7.3, particularly Definition 7.5 and Theorem 7.6.

Our linear optimal bound of the 2-handle complexity in Theorem 3.9 may be compared with a result of Costantino and Thurston [20] that a closed 3-manifold (which is not over a group) of complexity n bounds a 4-manifold whose complexity is bounded by $O(n^2)$.

Theorem 3.9 plays an essential role in the proofs of the explicit estimates of the Cheeger-Gromov bound in Theorems 1.5, 1.8, and 1.9. Briefly, we compute the Cheeger-Gromov invariants of a given 3-manifold M by using bordism W obtained by applying Theorem 3.9, and by controlling the 2-handle complexity of W efficiently, we obtain the explicit universal bounds. For this purpose, we need a chain level analogue u of W required in Theorem 3.9, and more importantly, we need to control the diameter of u. We do this by applying a general algebraic topological idea discussed in the next subsection.

1.5 Controlled Chain Homotopy

The second key ingredient of the proofs of Theorems 1.5, 1.8, and 1.9 is a method to estimate of the size of certain chain homotopies. It is best described using a notion of *controlled chain homotopy*, which we introduce in this subsection. It seems to be an interesting algebraic topological notion on its own, which may be compared with the topological notion of controlled homotopy. Readers primarily interested in controlled chain homotopy may first read this subsection and then proceed to Section 4.

We begin with basic definitions. Recall that the diameter $d(u)$ of a chain u is defined to be its L^1-norm (see Definition 1.17). As a convention, we assume that a chain complex C_* is positive, namely $C_i = 0$ for $i < 0$.

DEFINITION 1.18. Suppose C_* and D_* are based chain complexes, and $P : C_* \to D_{*+1}$ is a chain homotopy. We define the *diameter function* $d_P : \mathbb{Z} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ of P by

$$d_P(k) := \max\{d(P(c)) \mid c \in C_i \text{ is a basis element, } i \leq k\}.$$

For a partial chain homotopy P defined on C_i for $i \leq N$ only, we define $d_P(k)$ for $k \leq N$ exactly in the same way.

Let δ be a function from the domain of d_P to $\mathbb{Z}_{\geq 0}$. We say that P is a *δ-controlled* (*partial*) *chain homotopy* if $d_P(k) \leq \delta(k)$ for each k in the domain of d_P.

Note that $d_P(k)$ may be infinity in general. If P is a (partial) chain homotopy defined on a finitely generated chain complex, then $d_P(k)$ is finite whenever defined.

DEFINITION 1.19. Suppose $\mathcal{S} = \{P_A : C_*^A \to D_{*+1}^A\}_{A\in\mathcal{I}}$ is a collection of chain homotopies or a collection of partial chain homotopies defined in dimensions $\leq n$ for some fixed n. We say that $\mathcal{S}$ is *uniformly controlled by* δ if each P_A is a δ-controlled (partial) chain homotopy. The function δ is called a *control function* for $\mathcal{S}$.

Our focus is to understand how various families of chain homotopies can be uniformly controlled. A few additional words might make it clearer. In many cases the conclusion of a theorem on chain complexes can be understood as the existence of a certain chain homotopy, and in addition, such a theorem usually holds for a collection of objects, so that it indeed gives a family of chain homotopies indexed by the objects. For example, the classical Eilenberg-Zilber theorem says that $C_*(X \times Y)$ and $C_*(X) \otimes C_*(Y)$ are chain homotopy equivalent, that is, for *every* (X, Y) there are *chain homotopies* which tells us that the chain complexes are chain homotopy equivalences. Are these chain homotopies indexed by (X, Y) uniformly controlled?

In general, we consider the following metaquestion:

QUESTION 1.20. Pick a theorem about chain complexes or their homology. In the case of based chain complexes or their homology, can the theorem be understood in terms of uniformly controlled chain homotopies? If so, find (an estimate of) a control function.

In this paper, we observe several interesting cases for which a family of uniformly controlled chain homotopies exists, and we analyze the control functions in detail.

Our first theorem concerns the acyclic model theorem of Eilenberg and MacLane, which gives a family of functorial chain homotopies. As a fundamental observation, we show that if we use finitely many models in each dimension, then there is a single control function δ such that all the resulting functorial chain homotopies obtained by an acyclic model argument are uniformed controlled by δ. This result, which we call a *controlled acyclic model theorem*, is stated as Theorem 4.3. We discuss more details in Section 4.1.

As an application, we apply the controlled acyclic model theorem to products. In Section 4.2, we consider simplicial sets and the Moore complexes of the associated freely generated simplicial abelian groups as a general setup for products and based chain complexes. We present a *controlled Eilenberg-Zilber theorem*, which essentially says that the chain homotopy equivalence between the chain complex of a product and the tensor products of chain complexes can be understood in terms of uniformly controlled functorial chain homotopies. See Theorem 4.4 for more details.

We also consider the context of group homology. Recall that conjugation on a group induces the identity on the homology with integral coefficients. We give a quantitative generalization of this in terms of controlled chain homotopies. For a precise statement and related discussions, see Theorem 4.7 and Section 4.3.

We give another uniformly controlled chain homotopy result, concerning the result of Baumslag, Dyer, and Heller [3] that was already mentioned as a key ingredient of our topological proof of the existence of the Cheeger-Gromov bound (Theorem 1.3): there is a functorial embedding, say $i_G : G \hookrightarrow \mathcal{A}(G)$, of a group G to an acyclic group $\mathcal{A}(G)$ for each group G. From the viewpoint of controlled chain homotopy, the following natural question arises: for each G, is there a chain homotopy between the chain maps induced by the identity $\mathrm{id}_{\mathcal{A}(G)}$ and the trivial endomorphism of $\mathcal{A}(G)$ that forms a uniformly controlled family?

We give a partial answer. In [3], for each $n \geq 1$, they constructed a functorial embedding that we denote by $i_G^n : G \to \mathbb{A}^n(G)$, which induces a zero map $H_i(G;\Bbbk) \to H_i(\mathbb{A}^n(G);\Bbbk)$ for $1 \leq i \leq n$ and any field $\Bbbk$. (See Definition 5.1 for a precise description of $\mathbb{A}^n(G)$.) This may be viewed as an approximation of a functorial embedding into acyclic groups up to dimension n; in fact, it turns out that $\varinjlim \mathbb{A}^n(G)$ is acyclic and G embeds into it functorially. The following result is a controlled chain homotopy generalization of the homological property of i_G^n.

THEOREM (Theorem 5.2). *For each n, there is a family $\{\Phi_G^n \mid G \text{ is a group}\}$ of partial chain homotopies Φ_G^n defined in dimension $\leq n$ between the chain maps induced by the trivial map $e : G \to \mathbb{A}^n(G)$ and the embedding $i_G^n : G \to \mathbb{A}^n(G)$, which is uniformly controlled by a function δ_{BDH}. For $k \leq 4$, the value of $\delta_{\mathrm{BDH}}(k)$ is as follows*:

k	0	1	2	3	4
$\delta_{\mathrm{BDH}}(k)$	0	6	26	186	3410

Our proof of Theorem 5.2 consists of a careful construction of the chain homotopy Φ_G^n and its diameter estimate, using the above results on the acyclic model theorem and conjugation. We provide more detailed discussions and proofs in Section 5.

We remark that Theorem 5.2 for $n = 3$ (together with $\delta_{\mathrm{BDH}}(3) = 186$) is sufficient for our proofs of the Cheeger-Gromov bound estimates for 3-manifolds. See Section 6 for more details.

Organization of the Paper

In Section 2, we review the L^2-signature approach to the Cheeger-Gromov ρ-invariant and give a proof of Theorem 1.3. In Section 3, we give a construction of 4-dimensional bordisms and estimate the 2-handle complexity to prove Theorem 1.16. In Section 4, we develop the basic theory of controlled chain homotopy, including a controlled acyclic model theorem. In Section 5, we present a chain level approach to the result of Baumslag-Dyer-Heller. In Section 6 we obtain explicit estimates for the Cheeger-Gromov universal bound by proving Theorems 1.5, 1.8, and 1.9. In Section 7 we discuss the application to the complexity of 3-manifolds and prove that our linear Cheeger-Gromov bounds and geometric construction of efficient bordisms are asymptotically optimal. In the appendix, we discuss basic definitions and facts on simplicial sets and simplicial classifying spaces that we use in this paper for the reader's convenience.

2 Existence of Universal Bounds

In this section we give a topological proof of the existence of a universal bound for the Cheeger-Gromov invariant $\rho^{(2)}(M, \phi)$.

2.1 A Topological Definition of the Cheeger-Gromov ρ-Invariant

We begin by recalling a known topological definition of $\rho^{(2)}(M, \phi)$. We follow the approach introduced by Chang and Weinberger [12]; see also Harvey's work [26].

Suppose M is a closed topological $(4k-1)$-manifold, and $\phi : \pi_1(M) \to G$ is a homomorphism. When X is not path connected, as a convention, we denote by $\pi_1(X)$ the free product (= coproduct) $\coprod_\alpha \pi_1(X_\alpha)$ of the fundamental groups of the path components X_α of X. Suppose W is a $4k$-manifold with $\partial W = rM$, r disjoint copies of M. Suppose there are a monomorphism $G \hookrightarrow \Gamma$ and a homomorphism $\pi_1(W) \to \Gamma$ that make the following diagram commute:

(2.1)
$$\begin{tikzcd}
\coprod^{r} \pi_1(M) = \pi_1(rM) \arrow[r, "\coprod\phi"] \arrow[d, "i_*"'] & G \arrow[d, hook, dotted] \\
\pi_1(W) \arrow[r, dotted] & \Gamma
\end{tikzcd}$$

For a (discrete) group Γ, the *group von Neumann algebra* $\mathbb{N}\Gamma$ is defined as an algebra over $\mathbb{C}$ with involution. Lück's book [36] is a useful general reference on $\mathbb{N}\Gamma$; see also his paper [35]. In this paper we need the following known facts on $\mathbb{N}\Gamma$: (i) $\mathbb{C}\Gamma \subset \mathbb{N}\Gamma$ as a subalgebra. Consequently, in our case, $\mathbb{N}\Gamma$ is a local coefficient system over W via $\mathbb{C}[\pi_1(W)] \to \mathbb{C}\Gamma \subset \mathbb{N}\Gamma$. The homology $H_*(W; \mathbb{N}\Gamma)$ is defined as usual, and by Poincaré duality, the intersection form

$$\lambda : H_{2k}(W; \mathbb{N}\Gamma) \times H_{2k}(W; \mathbb{N}\Gamma) \to \mathbb{N}\Gamma$$

is defined. (ii) $\mathbb{N}\Gamma$ is semihereditary, that is, any finitely generated submodule of a finitely generated projective module over $\mathbb{N}\Gamma$ is projective; consequently, in our case, $H_{2k}(W; \mathbb{N}\Gamma)$ is a finitely generated module over $\mathbb{N}\Gamma$. (iii) For any hermitian form over a finitely generated $\mathbb{N}\Gamma$-module, there is a spectral decomposition; in our case, for the intersection form λ, we obtain an orthogonal direct sum decomposition

(2.2) $$H_{2k}(W; \mathbb{N}\Gamma) = V_+ \oplus V_- \oplus V_0$$

such that λ is zero, positive definite, and negative definite on V_0, V_+, and V_-, respectively; the positive definiteness means that $\lambda(x,x) = a^*a$ for some nonzero $a \in \mathbb{N}G$ whenever $x \in V_+$ is nonzero. (iv) There is a dimension function

$$\dim_\Gamma^{(2)} : \{\text{finitely generated } \mathbb{N}\Gamma\text{-modules}\} \to \mathbb{R}_{\geq 0}$$

that is additive for short exact sequences and satisfies $\dim_\Gamma^{(2)}(\mathbb{N}\Gamma) = 1$.

The L^2*-signature* of W over Γ is defined to be

$$\operatorname{sign}_\Gamma^{(2)} W = \dim_\Gamma^{(2)} V_+ - \dim_\Gamma^{(2)} V_-.$$

Now the L^2 ρ*-invariant* of (M,ϕ) is defined to be the signature defect

$$\rho^{(2)}(M,\phi) = \frac{1}{r}\bigl(\operatorname{sign}_\Gamma^{(2)} W - \operatorname{sign} W\bigr) \tag{2.3}$$

where $\operatorname{sign} W$ denotes the ordinary signature of W.

It is known that this topological definition of $\rho^{(2)}(M,\phi)$ is equivalent to the definition of Cheeger and Gromov given in [13] in terms of η-invariants. The proof depends on the L^2-index theorem for manifolds with boundary [13, 46] and the fact that various definitions of L^2-signatures are equivalent [37]. We remark that Cochran and Teichner present an excellent introduction to the analytic definition of $\rho(M,\phi)$ in [19, sec. 2].

Although the L^2-signature defect definition involves the bounding manifold W (and the enlargement Γ of the given G), it is known that a topological argument using bordism theory shows that such a W always exists and that $\rho^{(2)}(M,\phi)$ in (2.3) is independent of the choice of W, without appealing to analytic index theory. To the knowledge of the author, this method for the L^2-case first appeared in [12]. Since it is closely related to our techniques for the universal bound of the ρ-invariants that will be discussed in later sections, we give a proof below, without claiming any credit.

For the existence of W, we use a result of Kan and Thurston [31] that a group G embeds into an acyclic group, say Γ. Denote by Ω_*^{STOP} and $\Omega_*^{\mathrm{STOP}}(X)$ the oriented topological cobordism and bordism groups. By the foundational work of Kirby-Siebenmann [33] and Freedman-Quinn [23], $\Omega_*^{\mathrm{STOP}}(X)$ is a generalized homology theory. Since $H_p(\Gamma) = 0$ for $p \neq 0$, all the E^2 terms of the Atiyah-Hirzebruch spectral sequence

$$E^2_{pq} = H_p(\Gamma) \otimes \Omega_q^{\mathrm{STOP}} \Longrightarrow \Omega_n^{\mathrm{STOP}}(B\Gamma)$$

vanish except $E^2_{0,n} = \Omega_n^{\mathrm{STOP}}$. It follows that the inclusion $\{*\} \hookrightarrow B\Gamma$ induces an isomorphism $\Omega_n^{\mathrm{STOP}} \cong \Omega_n^{\mathrm{STOP}}(B\Gamma)$. Since $\Omega_{4k-1}^{\mathrm{STOP}} \otimes \mathbb{Q} \cong \Omega_{4k-1}^{\mathrm{SO}} \otimes \mathbb{Q} = 0$ due to Thom's classical work [48], it follows that rM bounds a $4k$-manifold W over $B\Gamma$ for some $r > 0$. This gives us the diagram (2.1).

For the independence of the choice of W, suppose the diagram (2.1) is also satisfied for (W', r', Γ') in place of (W, r, Γ). By L^2-induction (see, e.g., [13, eq. (2.3)], [36, p. 253], [17, prop. 5.13]), $\operatorname{sign}_\Gamma^{(2)}$ is left unchanged when Γ is replaced by another group containing Γ as a subgroup. Thus we may assume that $\Gamma = \Gamma'$ by replacing Γ and Γ' with the amalgamated product of them over G, and furthermore we may assume that Γ is acyclic using Kan-Thurston. Let $V = r'W \cup_{rr'M} -rW'$. Then V is a closed $4k$-manifold over Γ. Since Γ is acyclic, $\Omega_{4k}^{\text{STOP}} \cong \Omega_{4k}^{\text{STOP}}(B\Gamma)$, and therefore V is bordant to another V' that is over $B\Gamma$ via a constant map. We have $\operatorname{sign}_\Gamma^{(2)} V' = \operatorname{sign} V'$. Using Novikov additivity and that $\operatorname{sign}^{(2)}$ and sign are bordism invariants, we obtain

$$\frac{1}{r}\big(\operatorname{sign}_\Gamma^{(2)} W - \operatorname{sign} W\big) - \frac{1}{r'}\big(\operatorname{sign}_\Gamma^{(2)} W' - \operatorname{sign} W'\big) = \\ \frac{1}{rr'}\big(\operatorname{sign}_\Gamma^{(2)} V - \operatorname{sign} V\big) = \frac{1}{rr'}\big(\operatorname{sign}_\Gamma^{(2)} V' - \operatorname{sign} V'\big) = 0.$$

We remark that we may assume the codomain G of $\phi : \pi_1(M) \to G$ is countable. In fact, by L^2-induction, $\rho^{(2)}(M,\phi)$ is left unchanged when G is replaced by the countable group $\phi(\pi_1(M))$.

2.2 Existence of a Universal Bound

In this subsection we give a new proof of the existence of the Cheeger-Gromov universal bound, which applies directly to topological manifolds. Recall Theorem 1.3 from the introduction: *for any closed topological* $(4k-1)$*-manifold* M*, there is a constant* C_M *such that* $|\rho^{(2)}(M,\phi)| \le C_M$ *for any homomorphism* ϕ *of* $\pi_1(M)$.

In proving this using the topological definition of the Cheeger-Gromov invariants in Section 2.1, it is crucial to understand the "size" of the bounding $4k$-manifold W, since $\rho^{(2)}(M,\phi)$ is given by the L^2-signature defect of W as in (2.3). The key difficulty that is well known to experts is that the $4k$-manifold W in Section 2.1 depends on $\phi : \pi_1(M) \to G$ in general, since W is obtained by appealing to bordism theory over an acyclic group Γ, which depends on the group G.

We resolve this difficulty by employing the following *functorial* embedding of groups into acyclic groups, which was given by Baumslag, Dyer, and Heller.

THEOREM 2.1 (Baumslag-Dyer-Heller [3, theorem 5.5]). *There exists a functor* $\mathcal{A} : \mathbf{Gp} \to \mathbf{Gp}$ *on the category* $\mathbf{Gp}$ *of groups and a natural transformation* $\iota : \mathrm{id}_{\mathbf{Gp}} \to \mathcal{A}$ *such that* $\mathcal{A}(G)$ *is acylic and* $\iota_G : G \to \mathcal{A}(G)$ *is injective for any group* G.

We remark that $\mathcal{A}(G)$ given in [3] has the same cardinality as G if G is infinite and is generated by $(n+5)$ elements if G is generated by n elements.

PROOF OF THEOREM 1.3. Consider $\iota_{\pi_1(M)} : \pi_1(M) \to \mathcal{A}(\pi_1(M))$ given by Theorem 2.1. Since $\mathcal{A}(\pi_1(M))$ is acyclic, there is a $4k$-manifold W bounded by rM over $\mathcal{A}(\pi_1(M))$ for some $r > 0$, by the bordism argument in Section 2.1. Suppose $\phi : \pi_1(M) \to G$ is arbitrarily given. Let $\Gamma := \mathcal{A}(G)$. Then we have the following commutative diagram, by the functoriality of $\mathcal{A}$:

$$\begin{tikzcd}
\coprod^r \pi_1(M) \arrow[rrr, "\coprod \phi"] \arrow[dr, hook, "\coprod \iota_{\pi_1(M)}"] \arrow[dd, "i_*"'] & & & G \arrow[dr, hook, "\iota_G"] & \\
& \mathcal{A}(\pi_1(M)) \arrow[rrr, "\mathcal{A}(\phi)"] & & & \mathcal{A}(G) = \Gamma \\
\pi_1(W) \arrow[ur] & & & &
\end{tikzcd}$$

From this it follows that we can define $\rho^{(2)}(M,\phi)$ as the L^2-signature defect of W over Γ, as in (2.3). Note that our W is now independent of the choice of ϕ.

Recall that W has the homotopy type of a finite CW complex. Let $C_*(W;\mathbb{N}\Gamma)$ be the cellular chain complex defined using this CW structure. For N the number of $2k$-cells, we have $C_{2k}(W;\mathbb{N}\Gamma) \cong (\mathbb{N}\Gamma)^N$. By the additivity of the L^2-dimension under short exact sequences, we have

$$\begin{aligned}
\left|\operatorname{sign}^{(2)}_\Gamma W\right| &\le \dim^{(2)}_\Gamma V_+ + \dim^{(2)}_\Gamma V_- \\
&\le \dim^{(2)}_\Gamma H_{2k}(W;\mathbb{N}\Gamma) \le \dim^{(2)}_\Gamma C_{2k}(W;\mathbb{N}\Gamma) = N.
\end{aligned}$$

A similar argument shows that $|\operatorname{sign} W| \le N$. It follows that $|\rho^{(2)}(M,\phi)| \le 2N$ by (2.3). This completes the proof, since W, and consequently N, are independent of the choice of ϕ and G. □

3 Construction of Bordisms and 2-Handle Complexity

In this section, we introduce a general geometric construction that relates chain level algebraic data to a 4-dimensional bordism of a given 3-manifold. It may be viewed as a geometric incarnation of the Atiyah-Hirzebruch bordism spectral sequence. Furthermore, we give a more thorough analysis to obtain an explicit relationship between the complexity of the given algebraic data and the number of the 2-handles of an associated 4-dimensional bordism.

The results in this section will be used to reduce the problem of finding a universal bound for the ρ-invariants to a study of algebraic topological chain level information.

3.1 Geometric Construction of Bordisms

We begin with a straightforward observation on the Atiyah-Hirzebruch bordism spectral sequence, which is stated as Lemma 3.2 below. In this and the following sections, we consider the category of spaces X endowed with a map $\phi \colon X \to K$, where K is a fixed connected CW complex. We say that X *is over* K. If $K = B\Gamma$ for a group Γ, we say that X *is over* Γ. In this case we often view $\phi \colon X \to K$ as $\phi \colon \pi_1(X) \to \Gamma$ and vice versa.

We say that X is *trivially over* K if X is endowed with a constant map to K.

DEFINITION 3.1. A bordism W with $\partial W = M \sqcup -N$ over K is called a *bordism between* M *and a trivial end* if N is trivially over K.

LEMMA 3.2. *For a closed* 3*-manifold* M *endowed with* $\phi : M \to K$*, the following are equivalent*:

(1) M *bounds a smooth* 4*-manifold* V *over* K.
(2) *There is a smooth bordism* W *over* K *between* M *and a trivial end.*
(3) *The image* $\phi_*[M]$ *of the fundamental class* $[M] \in H_3(M)$ *is* 0 *in* $H_3(K)$.

PROOF. (1) implies (2) obviously. (2) implies (1) since $N := \partial W \setminus M$ bounds a 4-manifold that can be used to cap off W. From the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = H_p(K) \otimes \Omega^{\mathrm{SO}}_q \Longrightarrow \Omega^{\mathrm{SO}}_n(K)$$

and from that $\Omega^{\mathrm{SO}}_0 = \mathbb{Z}$, $\Omega^{\mathrm{SO}}_1 = \Omega^{\mathrm{SO}}_2 = \Omega^{\mathrm{SO}}_3 = 0$, it follows that $\Omega^{\mathrm{SO}}_3(K) \cong H_3(K)$ under the isomorphism sending the bordism class of $\phi : M \to K$ to $\phi_*[M] \in H_3(K)$. This shows that (1) is equivalent to (3). □

Remark 3.3. If (M, ϕ) is as in Lemma 3.2 and $K = B\Gamma$, then $\rho^{(2)}(M, \phi)$ can be defined as the L^2-signature defect of the bordism W in Lemma 3.2(2), as well as V in Lemma 3.2(1). For, if N is over Γ via ψ and $\partial W = M \sqcup -N$ over Γ, then $\rho^{(2)}(M, \phi) - \rho^{(2)}(N, \psi)$ is the L^2-signature defect of W by (2.3), and since the L^2-signature over a trivial map is equal to the ordinary signature, we have $\rho^{(2)}(N, \psi) = 0$ if ψ is trivial.

Suppose M is a closed 3-manifold equipped with a CW structure, whose 3-cells are oriented so that the sum ζ_M of the n-cells is a cycle representing the fundamental class $[M] \in H_n(M)$. We may assume that $\phi : M \to K$ is cellular by appealing to the cellular approximation theorem. Let $\phi_\#$ be the chain map on the cellular chain complex $C_*(-)$ induced by ϕ. Then we can restate Lemma 3.2(3) as follows:

LEMMA (Addendum to Lemma 3.2). $(3)'$ $\phi_\#(\zeta_M) = \partial u$ *for some* 4*-chain* u *in* $C_4(K)$.

The goal of this section is to discuss a more explicit relationship of the 4-dimensional bordism W in Lemma 3.2(2) and the 4-chain u in Lemma 3.2$(3)'$.

As an easier direction, if W is a bordism between M and a trivial end N, then for the sum ζ_W of oriented 4-cells of W that represent the fundamental class of $(W, \partial W)$, we have $\partial \zeta_W = \zeta_M - \zeta_N$. Since the image of ζ_N in $C_3(K)$ is 0, the image $u \in C_4(K)$ of ζ_W satisfies $\partial u = \phi_\#(\zeta_M)$.

For the converse, for a given 4-chain $u \in C_4(K)$ satisfying Lemma 3.2$(3)'$, we will present a construction of a bordism W between M and a trivial end. The rest of this subsection is devoted to this. This will tell us how the Atiyah-Hirzebruch spectral sequence is reinterpreted as a geometric construction and provide us the foundational idea of the more sophisticated analysis accomplished in Section 3.3.

To begin, as above, suppose a given closed 3-manifold M has a fixed CW complex structure, and $\phi : M \to K$ is cellular. Suppose $\phi_\#(\zeta_M) = \partial u$ for some $u \in C_4(K)$.

Our construction of W is based on the following observation. Let $K^{(i)}$ be the i-skeleton of K. By Atiyah-Hirzebruch, $\Omega_3^{\mathrm{SO}}(K)$ is filtered by

$$\Omega_3^{\mathrm{SO}}(K) = J_3 \supset J_2 \supset J_1 \supset J_0 \supset J_{-1} = 0$$

where $J_i = \mathrm{Im}\{\Omega_3^{\mathrm{SO}}(K^{(i)}) \to \Omega_3^{\mathrm{SO}}(K)\}$, and as in the proof of Lemma 3.2, we have

$$J_i/J_{i-1} \cong E^\infty_{i,3-i} \cong E^2_{i,3-i} = H_i(K) \otimes \Omega^{\mathrm{SO}}_{3-i} = \begin{cases} H_3(K) & \text{if } i = 3, \\ 0 & \text{if } i = 0, 1, 2. \end{cases} \tag{3.1}$$

Let $M_3 := M$. Obviously ϕ maps M_3 to $K^{(3)}$. For $i = 3$, (3.1) tells us that the existence of u implies that the bordism class of (M_3, ϕ) in $\Omega_3^{\mathrm{SO}}(K^{(3)})$ lies in the image of $\Omega_3^{\mathrm{SO}}(K^{(2)})$, that is, there is a bordism W_3 over K between M_3 and another 3-manifold, say M_2, such that M_2 maps to $K^{(2)}$. Similarly, for $i = 2$ and then for $i = 1$, (3.1) tells us that $\Omega^{\mathrm{SO}}_{3-i} = 0$ implies that M_i over $K^{(i)}$ admits a bordism W_i over K to another 3-manifold M_{i-1} that maps to $K^{(i-1)}$.

Once we have the bordisms W_i for $i = 3, 2, 1$, by concatenating them, we obtain a bordism W between the given M and the 3-manifold $N := M_0$. Since K is a connected CW complex, $N \to K^{(0)}$ is homotopic to a constant map. By altering the map $W \to K$ on a collar neighborhood of N using the homotopy, we may assume that N is over K via a constant map. This gives a desired bordism W between the given M and a trivial end N.

In Steps 1, 2, and 3 below, we present how to actually construct W_3, W_2, and W_1, using the given u and the facts $\Omega^{\mathrm{SO}}_{3-i} = 0$.

Step 1. REDUCTION TO THE 2-SKELETON $K^{(2)}$. We will construct W_3 using the given 4-chain u. Denote the characteristic map of a 4-cell e^4_α of K by $\phi_\alpha : D^4_\alpha \to K^{(4)}$ where D^4_α is a 4-disk. We may assume that the center of each 3-cell of K is a regular value of $\phi : M \to K^{(3)}$ and a regular value of each attaching map $\phi_\alpha|_{\partial D^4_\alpha} : \partial D^4_\alpha \to K^{(3)}$. Write the 4-chain u as $u = -\sum_\alpha n_\alpha e^4_\alpha$, and consider the 4-manifold $X = M \times [0,1] \sqcup \bigsqcup_\alpha n_\alpha D^4_\alpha$. View X as a bordism over K between $M \times 0$ and $M' := \partial X \setminus M \times 0$ via the map $X \to K$ induced by ϕ composed with the projection $M \times [0,1] \to M$ and the maps ϕ_α. Let $\psi : M' \to K$ be its restriction. The relation $\phi_\#(\zeta_M) - \partial u = 0$ implies that for the center y of each 3-cell of K, the points in $\psi^{-1}(y) \in M'$ signed by the local degree are canceled in pairs. For each canceling pair, attach to X a 1-handle joining these; the attaching 0-sphere is framed by pulling back a fixed framing at the regular value y, as usual. Let W_3 be the resulting cobordism, which is from $M = M \times 0$ to another 3-manifold, say M_2. The map ψ induces a map $W_3 \to K^{(4)}$ that maps $M \sqcup M_2$ to $K^{(3)}$. In addition, the image of M_2 is disjoint from the centers of 3-cells in $K^{(3)}$. It follows that by a homotopy on a collar neighborhood, we may assume that M_2 is mapped to $K^{(2)}$. This completes Step 1, as summarized in the following diagram:

$$\begin{tikzcd}
& M = M_3 \arrow[r, hook] \arrow[d, "\phi"] & W_3 \arrow[dll, dotted] & M_2 \arrow[l, hook'] \arrow[d, "\phi_2"] \\
K^{(4)} & K^{(3)} \arrow[l, hook'] & & K^{(2)} \arrow[ll, hook']
\end{tikzcd}$$

Step 2. REDUCTION TO THE 1-SKELETON $K^{(1)}$. For the map $\phi_2 : M_2 \to K^{(2)}$ obtained above, we may assume that the center y of a 2-cell of $K^{(2)}$ is a regular value. Then $\phi_2^{-1}(y)$ is a disjoint union of framed circles in M_2. Take $M_2 \times [0, 1]$, and attach 2-handles along the components of the framed 1-manifold $\phi_2^{-1}(y) \times 1 \subset M_2$. This gives a 4-dimensional cobordism W_2 from $M_2 = M_2 \times 0$ to another 3-manifold M_1, and ϕ_2 extends to $W_2 \to K^{(2)}$. By the construction, the image of M_1 in $K^{(2)}$ is disjoint from the centers of 2-cells. Therefore by a homotopy we may assume that $W_2 \to K^{(2)}$ restricts to a map $\phi_1 : M_1 \to K^{(1)}$.

We remark that in the above argument, $\Omega_1^{\mathrm{SO}} = 0$ appears as the fact that a circle bounds a disk so that we can attach a 2-handle along a circle.

Step 3. REDUCTION TO THE 0-SKELETON $K^{(0)}$. For the map $\phi_1 : M_1 \to K^{(1)}$, we may assume that the center of each 1-cell of $K^{(1)}$ is a regular value of ϕ_1. Then $S := \phi_1^{-1}(\{\text{centers of 1-cells}\})$ is a framed 2-submanifold in M. Since there is a union of handlebodies, say R, bounded by S, we can do "surgery" along S. More precisely, we obtain the trace of surgery by attaching $R \times [-1, 1]$ to $M_1 \times [0, 1]$ along $S \times [-1, 1] =$ normal bundle of S in $M_1 \times 1$. Performing this for each 1-cell of $K^{(1)}$, we obtain a cobordism W_1 from $M_1 = M_1 \times 0$ to another 3-manifold M_0, which is endowed with an induced map $W_1 \to K^{(1)}$. Similarly to the above, since the image of M_0 in $K^{(1)}$ under this map is away from the centers of 1-cells, we may assume that M_0 is mapped to $K^{(0)}$, by a homotopy.

We remark that in the above argument $\Omega_2^{\mathrm{SO}} = 0$ is used to guarantee that the 2-manifold S bounds a 3-manifold R.

The following diagram summarizes the above construction:

$$\begin{tikzcd}
& M_3 \arrow[r, hook] \arrow[d, "\phi"] & W_3 \arrow[dll, dotted] & M_2 \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\phi_2"'] & W_2 \arrow[dl, dotted] & M_1 \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\phi_1"'] & W_1 \arrow[dl, dotted] & M_0 \arrow[l, hook'] \arrow[d, "\phi_0"'] \\
K^{(4)} & K^{(3)} \arrow[l, hook'] & & K^{(2)} \arrow[ll, hook'] & & K^{(1)} \arrow[ll, hook'] & & K^{(0)} \arrow[ll, hook']
\end{tikzcd}$$

Remark 3.4. The operation of "surgery along a surface S" in Step 3 above can be translated to standard handle attachments as follows. Let g_i be the genus of a component S_i of $S = \phi_1^{-1}(\{\text{centers of 1-cells}\})$, and R_i be a handlebody bounded by S_i. Viewing R_i as a 0-handle D^3 with g_i 1-handles $D^2_{ij} \times [-1,1]$ $(1 \le j \le g_i)$ attached, and then turning it upside down, we see that attaching $R_i \times [-1,1]$ along $S_i \times [-1,1]$ is equivalent to attaching $D^2_{ij} \times [-1,1]^2$ along $\partial D^2_{ij} \times [-1,1]^2$ as 2-handles, and then attaching $D^3 \times [-1,1]$ along $\partial D^3 \times [-1,1]$ as a 3-handle. It follows that the bordism W_1 in Step 3 above consists of $(g_1 + \cdots + g_r)$ 2-handles and r 3-handles, where r is the number of components of S. This observation will be useful in Section 3.3.

Remark 3.5. From Steps 1, 2, and 3 above and from Remark 3.4, we obtain a handle decomposition of the bordism W. However, the above construction that uses CW complexes does not give bounds on the number of handles of W. For instance, regarding 2-handles, if we write $s =$ the number of components of $\phi_2^{-1}(\{\text{centers of 2-cells}\})$, and if r and the g_i are as in Remark 3.4, then our W has $s+(g_1+\cdots+g_r)$ 2-handles. Transversality arguments do not provide any control on the number of components s and r and the genera g_i of the preimage; in fact, a homotopy can increase s, r, and g_i arbitrarily. In order to provide an efficient control, we will use a simplicial setup and perform a more sophisticated analysis in Section 3.3.

3.2 Simplicial-Cellular Approximations of Maps to Classifying Spaces

In this subsection, we discuss some geometric ideas that arise from elementary simplicial set theory for readers not familiar with simplicial sets. (We present a brief review of basic necessary facts on simplicial sets in the appendix for the reader's convenience.) These will be used in the next subsection in order to control the 2-handle complexity of a bordism W.

We first formally state a generalization of simplicial complexes and simplicial maps by extracting geometric properties of simplicial sets (and their geometric realizations) that we need.

DEFINITION 3.6. Let Δ^n be the standard n-simplex.

(1) A CW complex X is a *pre-simplicial-cell complex* if each n-cell is endowed with a characteristic map of the form $\Delta^n \to X$. In particular, an open n-cell is identified with the interior of Δ^n. Often we call an n-cell an *n-simplex*. Note that a simplicial complex is a pre-simplicial-cell complex in an obvious way.
(2) A cellular map $X \to Y$ between pre-simplicial-cell complexes X and Y is called a *simplicial-cellular map* if its restriction on an open k-simplex of X is a surjection onto an open ℓ-simplex of Y $(\ell \le k)$ that extends to an affine surjection $\Delta^k \to \Delta^\ell$ sending vertices to vertices.

(3) A pre-simplicial-cell complex X is a *simplicial-cell complex* if the attaching map $\partial\Delta^k \to X^{(k-1)}$ of every k-cell is simplicial-cellular. Here we view the simplicial complex $\partial\Delta^k$ as a pre-simplicial-cell complex.

By abuse of terminology, we do not distinguish a simplicial-cell complex from its underlying space, and similarly for simplicial and CW complexes.

We note that the composition of simplicial-cellular maps is simplicial-cellular.

As an example, a simplicial complex is a simplicial-cell complex, and a simplicial map between simplicial complexes is a simplicial-cellular map. More generally, simplicial sets give us simplicial-cell complexes. More precisely, a simplicial set has the geometric realization, which is a CW complex due to Milnor [44]; in fact, his proof shows that the geometric realization is a simplicial-cell complex in the sense of Definition 3.6. See Appendix A.1 for a more detailed discussion.

The following special case will play a key role in the next subsection. It is well known that for a group G a $K(G,1)$ space is obtained as the geometric realization of the simplicial classifying space, that is, the nerve of G (for example, see [25, p. 6], [49, p. 257]). From now on, we denote this $K(G,1)$ space by BG. By the above, BG is a simplicial-cell complex. We remark that BG is not necessarily a simplicial complex.

THEOREM 3.7 (Simplicial-Cellular Approximation of Maps to BG). *Suppose X is the geometric realization of a simplicial set. Then any map $X \to BG$ is homotopic to a simplicial-cellular map.*

In this paper, we will apply Theorem 3.7 to a simplicial complex X; we note that a simplicial complex gives rise to a simplicial set (by ordering the vertices).

Since the author did not find it in the literature, a proof of Theorem 3.7 is given in the appendix; see Proposition A.1.

Remark 3.8. Theorem 3.7 may be compared with the standard simplicial and cellular approximation theorems. The simplicial approximation respects the simplicial structure but requires a subdivision of the domain. On the other hand, the cellular approximation does not require a subdivision but does not respect simplicial structures. Theorem 3.7 respects the simplicial structures and requires no subdivision. The latter is an important feature too, since controlling the number of simplicies is essential for our purpose.

3.3 Estimating the 2-Handle Complexity

In this subsection we present a simplicial refinement of the transversality-and-surgery arguments used in Section 3.1 and find an upper bound of the 2-handle complexity of the resulting bordism.

We define the *complexity of a triangulated* 3*-manifold* to be the number of 3-simplices. (Note that this is different from the notion of the (simplicial) complexity of a 3-manifold.) Recall from the introduction that the 2-handle complexity of a 4-dimensional bordism W is the minimal number of 2-handles in a handle decomposition of W.

For a triangulated closed 3-manifold M, let ζ_M be the sum of oriented 3-simplices of M that represents the fundamental class, as we did for a CW complex structure. Recall that the diameter $d(\zeta_M)$ is equal to the complexity of the triangulation.

The main result of this subsection is the following.

THEOREM 3.9. *Suppose M is a closed triangulated 3-manifold with complexity $d(\zeta_M)$. Suppose M is over a simplicial-cellular complex K via a simplicial-cellular map $\phi : M \to K$. If there is a 4-chain $u \in C_4(K)$ satisfying $\partial u = \phi_\#(\zeta_M)$, then there exists a smooth bordism W between M and a trivial end whose 2-handle complexity is at most $195 \cdot d(\zeta_M) + 975 \cdot d(u)$.*

We remark that when $K = B\Gamma$, any map $\phi : M \to K$ may be assumed to be a simplicial-cellular map up to homotopy, by Theorem 3.7.

Recall that in Section 3.1 we constructed a bordism W between M and a trivial end by stacking bordisms W_3, W_2, and W_1 such that $\partial W_i = M_i \sqcup -M_{i-1}$ over K, where $M_3 := M$ is the given 3-manifold, and M_i is over K via a map $\phi_i : M_i \to K^{(i)}$ to the i-skeleton for each i. The main strategy of our proof of Theorem 3.9 is to refine the construction of the W_i carefully to control the number of 2-handles. For this purpose, we will triangulate M_i and make ϕ_i simplicial-cellular. For the initial case, $M_3 = M$ is triangulated and $\phi_3 = \phi$ is simplicial-cellular by the hypothesis of Theorem 3.9. Arguments for W_i and M_{i-1} for $i = 3, 2, 1$ are given as the three propositions below.

PROPOSITION 3.10 (Step 1: Reduction to $K^{(2)}$ and Complexity Estimate). *Suppose M, ϕ, u are as in Theorem 3.9. Then there is a triangulated 3-manifold M_2 with complexity at most $n_2 := 18 \cdot d(\zeta_M) + 90 \cdot d(u)$, which is over K via a simplicial-cellular map $\phi_2 : M_2 \to K^{(2)}$, and there is a bordism W_3 over K between M and M_2 which has no 2-handles.*

PROOF. Following Step 1 in Section 3.1, we write $u = -\sum_\alpha n_\alpha \sigma_\alpha^4$, where the σ_α^4 are 4-simplices of K with attaching maps $\phi_\alpha : \partial\Delta_\alpha^4 \to K^{(3)}$. Here Δ_α^4 is a standard 4-simplex. Let $X := (M \times [0,1]) \sqcup (\bigsqcup_\alpha n_\alpha \Delta_\alpha^4)$. The 4-manifold X is a bordism over K between $M = M \times 0$ and $M' := (M \times 1) \sqcup (\bigsqcup_\alpha n_\alpha \partial\Delta_\alpha^4)$ via the map $X \to K$ induced by ϕ and the ϕ_α. Let $\psi : M' \to K$ be the restriction. The 3-manifold M' is triangulated using the given triangulation of M and the standard triangulation of $\partial\Delta_\alpha^4$. The map ψ is simplicial-cellular since ϕ and the ϕ_α are simplicial-cellular. From the relation $\phi_\#(\zeta_M) - \partial u = 0$, it follows that the 3-simplices of M' whose image under ψ is nonzero in $C_3(K)$ are canceled in pairs in the image under ψ. For each canceling pair of 3-simplices of M', we attach a 1-handle to X which joins their barycenters. To do it simplicially, we subdivide relevant 3-simplices as follows.

Recall that the product $\Delta^2 \times [0,1]$ is triangulated by a prism decomposition; see Figure 3.1. More precisely, ordering vertices of Δ^2 as $\{u_0, u_1, u_2\}$ and vertices of $[0,1]$ as $\{w_0, w_1\}$ and letting $v_{ij} = (u_i, w_j) \in \Delta^2 \times [0,1]$, the standard

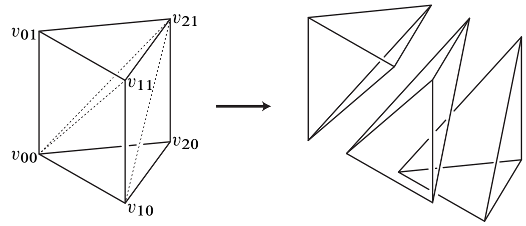


FIGURE 3.1. The standard prism decomposition of $\Delta^2 \times [0,1]$.

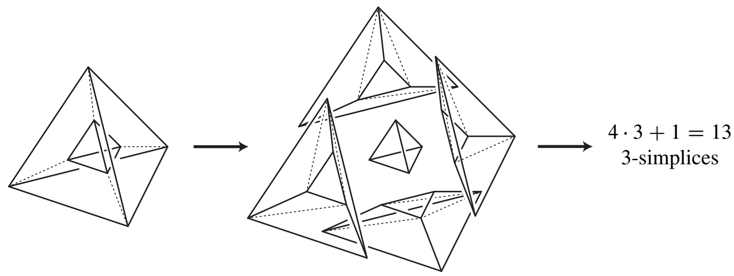


FIGURE 3.2. A subdivision of a 3-simplex for 1-handle attachment.

prism decomposition has 3-simplices $[v_{00}, v_{10}, v_{20}, v_{21}]$, $[v_{00}, v_{10}, v_{11}, v_{21}]$, and $[v_{00}, v_{01}, v_{11}, v_{21}]$. We note that we obtain several different prism decompositions by reordering vertices of Δ^2 and $[0,1]$.

Take a 3-simplex Δ' embedded in the interior of a standard 3-simplex Δ^3, and subdivide $\partial\Delta^3 \times [0,1] \cong \Delta^3 \setminus \operatorname{int}\Delta'$ by taking a prism triangulation of $\tau \times [0,1]$ for each face τ of Δ^3. As in Figure 3.2, one can choose prime decompositions appropriately in such a way that they agree on the intersections. This gives us a subdivision of Δ^3, which contains Δ' as a simplex. We call Δ' the *inner subsimplex* of this subdivision. We apply this subdivision to each 3-simplex of M' whose image under ψ is nonzero in $C_3(K)$, and then attach 1-handles $\Delta^3 \times [0,1]$ to X by identifying $\Delta^3 \times 0$ and $\Delta^3 \times 1$ with inner subsimplices of a canceling pair of 3-simplices. This gives a cobordism W_3 between $M = M_3$ and a new 3-manifold M_2 obtained from M' by surgery. By triangulating the belt tube $\partial\Delta^3 \times [0,1]$ of each 1-handle using a prism decomposition of (each face of Δ^3) $\times [0,1]$, and by combining it with the subdivision on M', we obtain a triangulation of M_2.

We want to show that there is a simplicial-cellular map $\phi_2 : M_2 \to K^{(2)}$ such that $\phi_3 \sqcup \phi_2 : M_3 \sqcup M_2 \to K$ extends to W_3. To do this explicitly, first observe that there is a map $\Delta^3 \to \Delta^3$ that is (i) simplicial with respect to the subdivision in Figure 3.2, (ii) collapses the collar $\Delta^3 - \operatorname{int}\Delta'$ onto $\partial\Delta^3$, (iii) stretches the inner subsimplex onto Δ^3, and (iv) is homotopic to the identity rel $\partial\Delta^3$. Composing it with the map $\psi : M' \to K$ on each subdivided 3-simplex on M, we obtain a simplicial-cellular map $\psi' : M' \to K$ with respect to the subdivision. Note

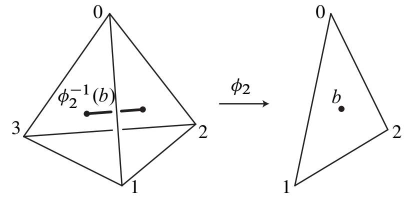


FIGURE 3.3. A simplicial projection $\Delta^3 \to \Delta^2$.

that ψ' is homotopic to ψ. Thus we may assume that the 4-manifold X is over K via a map $X \to K$ that restricts to ψ' on M'. Then $X \to K$ extends to the 1-handles and induces a map $W_3 \to K$, since the restrictions of ψ' on two inner subsimplices joined by a 1-handle are the same. Let $\phi_2 : M_2 \to K$ be the restriction. Since ψ' is simplicial-cellular, ϕ_2 is simplicial-cellular. Since ψ' sends $M' \setminus \bigsqcup$(inner simplices) to $K^{(2)}$, it follows that ϕ_2 sends M_2 to $K^{(2)}$. This completes the construction of the desired W_3, M_2, and $\phi_2 : M_2 \to K^{(2)}$.

Now we estimate the complexity of the triangulation of M_2. Let $n = d(\zeta_M)$, the complexity of the given triangulation of M. Since u has diameter $d(u) = \sum_\alpha |n_\alpha|$, the initial triangulation of $M' = (M\times 1)\sqcup(\bigsqcup_\alpha n_\alpha \partial\Delta^4_\alpha)$ has complexity $n+5d(u)$. Since our subdivision in Figure 3.2 produces 13 3-simplices from one 3-simplex, the complexity of the new subdivision of M' is at most $13(n+5d(u))$. The number of 1-handles attached is at most $(n + 5d(u))/2$, and each 1-handle attachment removes two 3-simplices (inner subsimplices) and adds $4 \cdot 3 = 12$ 3-simplices (those in the belt tube). Therefore, as claimed, the complexity of the triangulation of M_2 is at most

$$n_2 := 12(n + 5d(u)) + 12 \cdot \frac{n + 5d(u)}{2} = 18n + 90d(u).$$

From our construction, it is obvious that W has no 2-handles. □

PROPOSITION 3.11 (Step 2: Reduction to $K^{(1)}$ and Complexity Estimate). *Suppose M_2 is a closed triangulated 3-manifold with complexity n_2, which is over K via a simplicial-cellular map $\phi_2 : M_2 \to K^{(2)}$. Then there is another triangulated 3-manifold M_1 with complexity at most $n_1 := 21n_2$, which is over K via a simplicial-cellular map $\phi_1 : M_1 \to K^{(1)}$, and there is a bordism W_2 over K between M_2 and M_1 with 2-handle complexity at most $\lfloor n_2/3 \rfloor$.*

PROOF. To obtain W_2, we will attach 2-handles to $M_2 \times [0, 1]$ along the inverse image of the barycenter of each 2-simplex of K under ϕ_2, similarly to Step 2 of Section 3.1. Fix a 2-simplex of K and denote its barycenter by b. If the interior of a 3-simplex of M_2 meets $\phi_2^{-1}(b)$, then since ϕ_2 is a simplicial-cellular map, it follows that ϕ_2 on the 3-simplex is an affine projection $\Delta^3 \to \Delta^2$ onto the 2-simplex sending vertices to vertices; see Figure 3.3, which illustrates the case $[0, 1, 2, 3] \mapsto [0, 1, 2, 2]$. Figure 3.3 also shows the preimage $\phi_2^{-1}(b)$ in the 3-simplex.

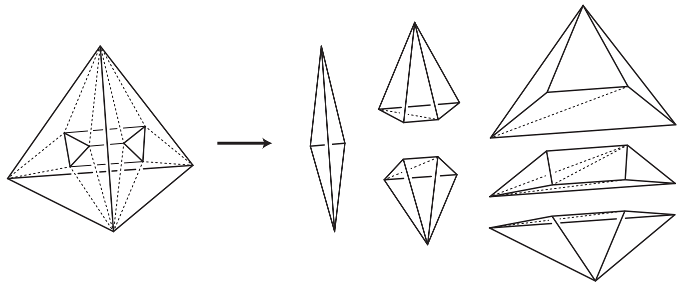


FIGURE 3.4. A subdivision of a 3-simplex with a triangular prism removed.

We take a sufficiently thin tubular neighborhood $U \cong \phi_2^{-1}(b) \times \Delta^2$ of $\phi_2^{-1}(b)$ in M_2 in such a way that the intersection of U and a 3-simplex of M_2 is a triangular prism or empty. We triangulate the exterior $M_2 \setminus \operatorname{int}(U)$ by subdividing each 3-simplex with a triangular prism removed as in Figure 3.4; we first decompose it into one 3-simplex, one triangular prism, and 4 quadrangular pyramids, and then divide the triangular prism and quadrangular pyramids along the dashed lines to obtain a subdivision with $1 + 3 + 4 \cdot 2 = 12$ 3-simplices. Since the subdivision of the two front faces of the original 3-simplex shown in the left of Figure 3.4 are identical and the two back faces are not subdivided, our subdivisions agree on the intersection of any two such 3-simplices. Observe that $\partial(M_2 \setminus \operatorname{int}(U)) = \partial U$ meets a 3-simplex of M_2 in three squares forming a cylinder as in Figure 3.4, where each square has been triangulated into two 2-simplices. For later use, we note that we can alter the triangulation of these squares by changing the subdivisions of the quadrangular pyramids and the triangular prism in Figure 3.4.

Now we consider 2-handle attachments. The preimage $\phi_2^{-1}(b) \subset M_2$ is a disjoint union of piecewise linear circles. Suppose C is a circle component of $\phi_2^{-1}(b)$. Let r be the number of 3-simplices of M_2 that C passes through as in the local picture shown in Figure 3.3, that is, C is an r-gon. Take a 2-handle $D \times \Delta^2$, where D is a 2-disk. Triangulate D into r triangles by drawing r line segments from the center to the perimeter, and then triangulate $D \times (\text{each face of } \Delta^2) \cong D \times [0, 1]$ by ordering the 0-simplices of D and then taking the prism decomposition of (each 2-simplex of D) $\times [0, 1]$. Gluing these, we obtain a triangulation of the belt tube $D \times \partial\Delta^2$ of the 2-handle. We attach the 2-handle $D \times \Delta^2$ to $M_2 \times [0, 1]$ by identifying the neighborhood $C \times \Delta^2 \subset M_2 = M_2 \times 1$ with the attaching tube $\partial D \times \Delta^2$. We may assume that the triangulation of $\partial D \times \partial\Delta^2$ agrees with that of $\partial(M \setminus \operatorname{int}(U))$ by altering the latter as mentioned above if necessary. We note that our triangulation of the belt tube of this 2-handle has $3 \cdot 3r = 9r$ 3-simplices.

Attaching 2-handles for each 2-simplex of K in this way, we obtain a cobordism W_2 between M_2 and another 3-manifold M_1, together with a triangulation of M_1.

We make W_2 a bordism over K similarly to Step 1 above: observe that there is a piecewise linear endomorphism of the 3-simplex Δ^3 shown in the left of Figure 3.4

that restricts to a simplicial-cellular map of the exterior $\Delta^3 \setminus \operatorname{int}(U)$ onto $A := \partial\Delta^3 \setminus$ (interior of the two faces of Δ^3 meeting $\phi^{-1}(b)$) and is homotopic to the identity rel A. From this it follows that the map $\phi_2 : M_2 \to K^{(2)}$ is homotopic to a map that restricts to a simplicial-cellular map $M_2 \setminus \operatorname{int}(U) \to K^{(1)}$ and extends to $W_2 \to K^{(2)}$. Also, $W_2 \to K^{(2)}$ restricts to a simplicial-cellular map $\phi_1 : M_1 \to K^{(1)}$. In particular, W_2 is a bordism over K between (M_2, ϕ_2) and (M_1, ϕ_1).

Now we estimate the complexity of M_1. Recall the hypothesis that M_2 has n_2 3-simplices. Our subdivision of $M \setminus \operatorname{int}(U)$ has at most $12n_2$ 3-simplices, since each 3-simplex that meets an attaching circle contributes 12 3-simplices as observed above (see Figure 3.4). Suppose we attach s 2-handles and the i^{th} 2-handle is attached along an r_i-gon. As observed above, the belt tube of the i^{th} 2-handle has $9r_i$ 3-simplices. Therefore our triangulation of M_1 has complexity at most $12n_2 + 9(r_1 + \cdots + r_s)$. Since each 3-simplex of M_2 can contribute at most one line segment to the attaching circles, we have $r_1 + \cdots + r_s \le n_2$. It follows that M_2 has complexity at most $21n_2$. Since $r_i \ge 3$, we also obtain that $3s \le n_2$ as claimed. $\square$

PROPOSITION 3.12 (Step 3: Reduction to $K^{(0)}$ and Complexity Estimate). *Suppose M_1 is a closed triangulated 3-manifold with complexity n_1 that is over K via a simplicial-cellular map $\phi_1 : M_1 \to K^{(1)}$. Then there is another 3-manifold M_0 that is over K via a map $\phi_0 : M_0 \to K^{(0)}$, and there is a bordism W_1 over K between M_1 and M_0 whose 2-handle complexity is at most $\lfloor n_1/2 \rfloor$.*

PROOF. We construct the bordism W_1 similarly to Step 3 of Section 3.1, namely by attaching $R_i \times [0,1]$ to $M_1 \times [0,1]$, where R_i is a handlebody bounded by a component S_i of the preimage of the barycenter of a 1-simplex of K under ϕ_1. Recall from Remark 3.5 that if S_i has genus g_i, then attaching R_i is equivalent to attaching g_i 2-handles and one 3-handle.

Since ϕ_1 is simplicial-cellular, the preimage $\phi_1^{-1}(b)$ of a barycenter b of a 1-simplex of K intersects a 3-simplex Δ^3 of M_1 as shown in Figure 3.5; we have two possibilities, where $\phi^{-1}(b) \cap \Delta^3$ is either a triangle or a quadrangle. By dividing each quadrangle in $\phi^{-1}(b)$ into two triangles, we obtain a triangulation of the 2-manifold $\phi_1^{-1}(b)$. Since M_1 has n_1 3-simplices and each 3-simplex can contribute at most two triangles to $\phi^{-1}(b)$, it follows that the 2-manifold $\bigsqcup_i S_i$ has a triangulation with at most $2n_1$ 2-simplices.

To estimate the genera, we invoke the following observation:

LEMMA 3.13. *A connected closed surface admitting a triangulation with n 2-simplices has genus at most $\lfloor \frac{n-2}{4} \rfloor$.*

PROOF. Since there are $\frac{3n}{2}$ 1-simplices, the Euler characteristic $2-2g$ is equal to $n - \frac{3n}{2} + v$, where v is the number of 0-simplices. Since $v \ge 3$, it follows that $g \le \frac{n-2}{4}$. $\square$

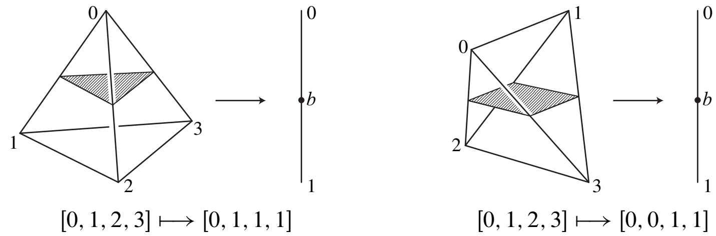


FIGURE 3.5. Simplicial projections $\Delta^3 \to \Delta^1$.

Returning to the proof of Proposition 3.12, suppose the inverse image of the union of the barycenters of 1-simplices of K under ϕ_1 has r components $S_1, \dots, S_r$, and suppose S_i has m_i 2-simplices in its triangulation. By Lemma 3.13, the genus g_i of S_i is at most $m_i/4$. Since $m_1 + \cdots + m_r \le 2n_1$, it follows that $g_1 + \cdots + g_r \le n_1/2$. Therefore, the 2-handle complexity of W_1 is at most $n_1/2$ as claimed. □

Now we combine the above three propositions to give a proof of Theorem 3.9.

PROOF OF THEOREM 3.9. Let $M_3 = M$ and $\phi_3 = \phi$, and apply Propositions 3.10, 3.11, and 3.12 to obtain bordisms W_3, W_2, and W_1 together with (M_2, ϕ_2), (M_1, ϕ_1), and (M_0, ϕ_0). Concatenating W_3, W_2, and W_1, we obtain a bordism W over K between M and $N := M_0$. Since ϕ_0 is a map to $K^{(0)}$, ϕ_0 is homotopic to a constant map, and so we may assume that N is trivially over K. By Propositions 3.10, 3.11, and 3.12, M_2 and M_1 have complexity at most $n_2 := 18n + 90d(u)$ and $n_1 := 21n_2 = 378n + 1890d(u)$, respectively. Also, W_3 has no 2-handles, W_2 has at most $n_2/3 = 6n + 30d(u)$ 2-handles, and W_1 has at most $n_1/2 = 189n + 945d(u)$ 2-handles. It follows that the 2-handle complexity of W is not greater than

$$6n + 30d(u) + 189n + 945d(u) = 195n + 975d(u). \qquad \square$$

In light of Theorem 3.9, finding a 4-chain u with controlled diameter $d(u)$ is essential in constructing an efficient 4-dimensional bordism to a trivial end. This will be done by using the results developed in the next section.

4 Controlled Chain Homotopy

In this section we develop some useful results on controlled chain homotopy. We recall basic definitions from the introduction. In this paper we assume that chain complexes are always positive. We also assume that chain complexes are over $\mathbb{Z}$, although everything holds over a ring R endowed with a norm $|\cdot|$. The *diameter* $d(u)$ of a chain u in a based chain complex is defined to be its L^1-norm, that is, if $u = \sum_\alpha n_\alpha e_\alpha$ where $\{e_\alpha\}$ is the given basis, then $d(u) = \sum_\alpha |n_\alpha|$. For a chain homotopy $P : C_* \to D_{*+1}$ between based chain complexes C_* and D_*, the *diameter function* d_P of P is defined by

$$d_P(k) := \max\{d(P(c)) \mid c \in C_i \text{ is a basis element}, i \le k\}.$$

If P is a partial chain homotopy that is defined on C_i for $i \le N$ only, then $d_P(k)$ is defined for $k \le N$. Note that $d_P(k)$ may not be finite if $\bigoplus_{i \le k} C_i$ is not finitely generated.

For a function δ from the domain of d_P to $\mathbb{Z}_{\ge 0}$, we say that P is a *δ-controlled (partial) chain homotopy* if $d_P(k) \le \delta(k)$ for each k.

Similarly to the chain homotopy case, the *diameter function* $d_\phi(k)$ of a chain map $\phi : C_* \to D_*$ is defined by

$$d_\phi(k) = \max\{d(\phi(u)) \mid u \in C_i \text{ is a basis element}, i \le k\}.$$

We say that a chain map $f : C_* \to D_*$ between based chain complexes C_* and D_* is *based* if f takes a basis element to a basis element. A based chain map ϕ has $d_\phi(k) = 1$.

For a chain homotopy or a chain map P, $d(P(z)) \le d_P(k) \cdot d(z)$ for any chain z of dimension at most k. We state a few more basic facts for later use:

LEMMA 4.1.

(1) (Sum) *If $P : \phi \simeq \psi$ and $Q : \zeta \simeq \xi$ for ϕ, ψ, ζ, $\xi : C_* \to D_*$, then $P + Q : \phi + \zeta \simeq \psi + \xi$ and $d_{P+Q}(k) \le d_P(k) + d_Q(k)$.*

(2) (Composition) *If $P : \phi \simeq \psi$ and $Q : \zeta \simeq \xi$ for chain maps $\phi, \psi : C_* \to D_*$ and $\zeta, \xi : D_* \to E_*$, then $\zeta P + Q\psi : \zeta\phi \simeq \xi\psi$ and $d_{\zeta P + Q\psi}(k) \le d_\zeta(k) \cdot d_P(k) + d_Q(k) \cdot d_\psi(k)$.*

(3) (Tensor product) *If $P : \phi \simeq \psi$ and $Q : \zeta \simeq \xi$ for chain maps $\phi, \psi : C_* \to D_*$ and $\zeta, \xi : C'_* \to D'_*$, then*

$$\Phi(\sigma \otimes \tau) := (P \otimes \zeta + (-1)^{|\sigma|}\psi \otimes Q)(\sigma \otimes \tau)$$

is a chain homotopy $\Phi : \phi \otimes \zeta \simeq \psi \otimes \xi$, and $d_\Phi(k) \le d_P(k) \cdot d_\zeta(k) + d_\psi(k) \cdot d_Q(k)$.

The analogues for partial chain homotopies hold too.

The proof of Lemma 4.1 is straightforward. We omit details.

From Definition 1.19 in the introduction, we recall the notion of a uniformly controlled family of chain homotopies: suppose $\mathcal{S} = \{P_A : C^A_* \to D^A_{*+1}\}_{A \in \mathcal{I}}$ is a collection of chain homotopies or a collection of partial chain homotopies defined in dimensions $\le n$ for some fixed n. We say that $\mathcal{S}$ is *uniformly controlled by δ* if each P_A is a δ-controlled chain homotopy.

In many cases a family of chain homotopies comes with functoriality in the following sense. Let $\mathbf{Ch}_+$ be the category of positive chain complexes over $\mathbb{Z}$; morphisms are degree 0 chain maps as usual. Suppose $\mathbf{C}$ is a category, $F, G : \mathbf{C} \to \mathbf{Ch}_+$ are functors, and $\phi, \psi : F \to G$ are natural transformations, that is, for each $A \in \mathbf{C}$ we have chain complexes $F(A)$, $G(A)$ and chain maps $\phi_A, \psi_A : F(A) \to G(A)$ that are functorial in A. We say that $\{P_A : \phi_A \simeq \psi_A\}_{A \in \mathbf{C}}$ is a *family of natural chain homotopies between ϕ and ψ* if $P_A : F(A)_* \to G(A)_{*+1}$

is functorial in A and $P_A\partial + \partial P_A = \psi_A - \phi_A$ for each $A \in \mathbf{C}$. The partial chain homotopy analogue is defined similarly.

We denote by $\mathbf{Ch}_+^b$ the category of positive based chain complexes and (not necessarily based) chain maps. The above paragraph applies to $\mathbf{Ch}_+^b$ similarly.

4.1 Controlled Acyclic Model Theorem

Our first source of a uniformly controlled family of natural chain homotopies is the classical acyclic model theorem of Eilenberg and MacLane [21].

We recall two basic definitions used to state the standard acyclic model theorem. We say that $F : \mathbf{C} \to \mathbf{Ch}_+$ (or $\mathbf{Ch}_+^b$) is *acyclic* with respect to a collection $\mathcal{M}$ of objects in $\mathbf{C}$ if the chain complex $F(A)$ is acyclic for each A in $\mathcal{M}$. Also, we say that F is *free* with respect to $\mathcal{M}$ if for each i there is a collection $\mathcal{M}_i = \{(A_\lambda, c_\lambda)\}_\lambda$ with $A_\lambda \in \mathcal{M}$ and $c_\lambda \in F(A_\lambda)_i$ such that for any object B in $\mathbf{C}$, $F(B)_i$ is a free abelian group and the elements $F(f)(c_\lambda) \in F(B)_i$ for $f \in \operatorname{Mor}(A_\lambda, B)$ are distinct and form a basis. We define analogues for based chain complexes:

DEFINITION 4.2.

(1) A functor $F : \mathbf{C} \to \mathbf{Ch}_+^b$ is *based* if for any $f \in \operatorname{Mor}_{\mathbf{C}}(A, B)$, $F(f) \in \operatorname{Mor}_{\mathbf{Ch}_+^b}(F(A), F(B))$ is a based chain map. Also, F is *based-acyclic* if F is based and acyclic.

(2) A functor $F : \mathbf{C} \to \mathbf{Ch}_+^b$ is *based-free* with respect to $\mathcal{M}$ if for each i there is a collection $\mathcal{M}_i = \{(A_\lambda, c_\lambda)\}_\lambda$ with $A_\lambda \in \mathcal{M}$ and $c_\lambda \in F(A_\lambda)_i$ such that for any $A \in \mathbf{C}$, the elements $F(f)(c_\lambda) \in F(A)_i$ for $f \in \operatorname{Mor}(A_\lambda, A)$ are distinct and form the preferred basis of the based free abelian group $F(A)_i$. In addition, if $\mathcal{M}_i$ is finite for each i, then we say that F is *finitely based-free*.

Observe that F is automatically based if F is based-free.

THEOREM 4.3 (Controlled Acyclic Model Theorem). *Suppose $F, G : \mathbf{C} \to \mathbf{Ch}_+^b$ are functors, F is finitely based-free with respect to $\mathcal{M}$, and G is based-acyclic with respect to $\mathcal{M}$. Then the following hold*:

(1) *Any natural transformation $\phi_0 : H_0 \circ F \to H_0 \circ G$ extends to a natural transformation $\phi : F \to G$.*

(2) *Suppose $\phi, \psi : F \to G$ are natural transformations that induce the same transformation $H_0 \circ F \to H_0 \circ G$. Then there exist a function $\delta : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ and a family of natural chain homotopies $\{P_A : \phi_A \simeq \psi_A\}$ that is uniformly controlled by δ.*

The key is that that even when the rank of the chain complexes is unbounded, we have a uniform control δ if there are only finitely many models in each dimension.

PROOF. Recall that (1) is a conclusion of a standard acyclic model argument.

For (2), recall the construction of a family of chain homotopies

$$P_A = \{(P_A)_i : F(A)_{i-1} \to G(A)_i\}, \quad A \in \mathbf{C},$$

from the standard acyclic model argument: assume $(P_A)_{i-1}$ has been defined. Using that $G(A_\lambda)$ is acyclic for each $(A_\lambda, c_\lambda) \in \mathcal{M}_i$, we obtain a chain, which we denote by $(P_{A_\lambda})_i(c_\lambda) \in G(A_\lambda)_{i+1}$ as abuse of notation for now, that makes the equation $P_{A_\lambda}\partial + \partial P_{A_\lambda} = \psi_{A_\lambda} - \phi_{A_\lambda}$ satisfied at $c_\lambda \in F(A_\lambda)_i$; then for an arbitrary $A \in \mathbf{C}$, using that F is free, we define $(P_A)_i$ on a basis element by $(P_A)_i(F(f)(c_\lambda)) := G(f)((P_{A_\lambda})_i(c_\lambda))$ and extend it linearly.

Since $G(f)$ is based, the diameter of $(P_A)_i(F(f)(c_\lambda))$ is equal to the diameter of $(P_{A_\lambda})_i(c_\lambda)$. Since $F(A)_i$ is based by $\{F(f)(c_\lambda)\}$, it follows that for any $A \in \mathbf{C}$ the diameter function d_{P_A} of P_A is equal to the function δ defined by

$$\delta(k) := \max\{d((P_{A_\lambda})_i(c_\lambda)) \mid i \le k,\ (A_\lambda, c_\lambda) \in \mathcal{M}_i\}.$$

The value $\delta(k)$ is finite for any k, since $\mathcal{M}_i$ is a finite collection for any i. □

The proof of Theorem 4.3 tells us that the control function $\delta(k)$ is obtained from the diameter of the chain homotopy on the models. Using this, we can often compute $\delta(k)$ explicitly, at least for small k. We deal with an example in the next subsection.

4.2 Controlled Eilenberg-Zilber Theorem

In this subsection, we investigate uniform control for the chain homotopies of the Eilenberg-Zilber theorem for products. Our result is best described using simplicial sets. Readers not familiar with simplicial sets may refer to our quick review of basic definitions in the appendix.

We first state a theorem, and then recall the terminologies used in the statement for the reader's convenience.

THEOREM 4.4 (Controlled Eilenberg-Zilber Theorem). *For simplicial sets X and Y, let*

$$\Delta_{X,Y} : C_*(X \times Y) \longrightarrow C_*(X) \otimes C_*(Y),$$
$$\nabla_{X,Y} : C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y),$$

be the Alexander-Whitney map and the shuffle map. Then there is a natural family of chain homotopies

$$\{P_{X,Y} : \nabla_{X,Y} \circ \Delta_{X,Y} \simeq \mathrm{id}_{C_*(X\times Y)} \mid X \text{ and } Y \text{ are simplicial sets}\}$$

that is uniformly controlled by a function $\delta_{\mathrm{EZ}}(k)$. Furthermore, the value of $\delta_{\mathrm{EZ}}(k)$ for $k \le 4$ is as follows:

k	0	1	2	3	4
$\delta_{\mathrm{EZ}}(k)$	0	1	4	11	26

Remark 4.5.

(1) Of course the existence of the chain homotopy $P_{X,Y}$ is due to Eilenberg-Zilber [21]. What Theorem 4.4 newly gives is that $\{P_{X,Y}\}$ is uniformly controlled, and that the values of the control function δ_{EZ} are as above.

(2) In our applications, explicit values of $\delta_{\mathrm{EZ}}(k)$ for $k \le 3$ are sufficient, since we are interested in chains arising from 3-manifolds.

Recall that a simplicial set X consists of sets X_n $(n = 0, 1, \dots)$, face maps $d_i : X_n \to X_{n-1}$, and degeneracy maps $s_i : X_n \to X_{n+1}$ $(i = 0, 1, \dots, n)$. (See the appendix for instance.) We call $\sigma \in X_n$ an *n-simplex* of X. Let $\mathbb{Z}X$ be the simplicial abelian group generated by X, and denote its (unnormalized) Moore complex by $\mathbb{Z}X_*$. In other words, $\mathbb{Z}X_n$ is the free abelian group generated by X_n, and the boundary map $\partial : \mathbb{Z}X_n \to \mathbb{Z}X_{n-1}$ is defined by $\partial\sigma = \sum_i (-1)^i d_i\sigma$ for $\sigma \in X_n$. We always view $\mathbb{Z}X_*$ as a based chain complex; each $\mathbb{Z}X_n$ is based by the n-simplices. We denote the homology by $H_*(X) := H_*(\mathbb{Z}X_*)$.

For two simplicial sets X and Y, the product $X \times Y$ is defined by $(X \times Y)_n := X_n \times Y_n$; writing $\sigma \times \tau := (\sigma, \tau) \in X_n \times Y_n$, the face and degeneracy maps are defined by $d_i(\sigma \times \tau) = d_i\sigma \times d_i\tau$ and $s_i(\sigma \times \tau) = s_i\sigma \times s_i\tau$.

The *Alexander-Whitney map*

$$\Delta = \Delta_{X,Y} : \mathbb{Z}(X \times Y)_* \longrightarrow \mathbb{Z}X_* \otimes \mathbb{Z}Y_*$$

is defined by

$$\Delta(\sigma \times \tau) = \sum_{i=0}^{n} d_{i+1} \cdots d_n\sigma \otimes (d_0)^i \tau \tag{4.1}$$

for $\sigma \times \tau \in X_n \times Y_n$. To define its chain homotopy inverse, we use the following notation. A *(p, q)-shuffle* $(\mu, \nu) = (\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_q)$ is a permutation of $(1, \dots, p + q)$ such that $\{\mu_i\}$, $\{\nu_i\}$ are both increasing. Let $\epsilon(\mu, \nu)$ be the sign of the permutation, and $S_{p,q}$ be the set of (p, q)-shuffles. Then the *shuffle map* (or the Eilenberg-Zilber map or the Eilenberg-MacLane map)

$$\nabla = \nabla_{X,Y} : \mathbb{Z}X_* \otimes \mathbb{Z}Y_* \longrightarrow \mathbb{Z}(X \times Y)_*$$

is defined by

$$\nabla(\sigma \otimes \tau) = \sum_{(\mu,\nu) \in S_{p,q}} (-1)^{\epsilon(\mu,\nu)} (s_{\nu_q} \cdots s_{\nu_1}\sigma) \times (s_{\mu_p} \cdots s_{\mu_1}\tau) \tag{4.2}$$

for $\sigma \otimes \tau \in \mathbb{Z}X_p \otimes \mathbb{Z}Y_q$.

It is verified straightforwardly that Δ and ∇ are chain maps and $\Delta \circ \nabla = \mathrm{id}$ on $\mathbb{Z}X_* \otimes \mathbb{Z}Y_*$. It is known that $\nabla \circ \Delta$ is chain homotopic to id on $\mathbb{Z}(X \times Y)_*$ by an acyclic model argument with $\mathcal{M} = \{\Delta^n \times \Delta^n \mid n \ge 0\}$ as models. By using our controlled version of the acyclic model theorem (Theorem 4.3), we can obtain the additional conclusions on the chain homotopy $\nabla \circ \Delta \simeq \mathrm{id}$ as stated in Theorem 4.4. We describe details below.

PROOF OF THEOREM 4.4. We follow the standard acyclic model argument for a product. Let **sSet** be the category of simplicial sets and define a functor F : $\mathbf{sSet} \times \mathbf{sSet} \to \mathbf{Ch}^b_+$ by $F(X,Y) := \mathbb{Z}(X \times Y)_*$. By definition, F is based. Let Δ^n be the standard n-simplex as a simplicial set; we write a k-simplex of Δ^n as a sequence $[v_0, \dots, v_k]$ of integers v_i such that $0 \le v_0 \le \cdots \le v_k \le n$. Let $\mathcal{M} = \{(\Delta^n, \Delta^n) \mid n \ge 0\}$. Then F is acyclic with respect to $\mathcal{M}$, since $\Delta^n \times \Delta^n$ is contractible. Also, F is finitely based-free with respect to $\mathcal{M}$ since $\mathbb{Z}(X \times Y)_n$ is freely generated by

$$\{f[0,\dots,n] \times g[0,\dots,n] \in (X \times Y)_n \mid \Delta^n \xrightarrow{f} X,\ \Delta^n \xrightarrow{g} Y \text{ are morphisms}\}.$$

Note that there is only one model (Δ^n, Δ^n) in each dimension n.

By Theorem 4.3, it follows that there is a function $\delta_{\mathrm{EZ}}(k)$ and a natural family of chain homotopies $P_{X,Y} : \mathbb{Z}(X \times Y)_* \to \mathbb{Z}(X \times Y)_{*+1}$ between $\nabla_{X,Y} \circ \Delta_{X,Y}$ and id, which is uniformly controlled by δ_{EZ}.

We will explicitly compute the value $\delta_{\mathrm{EZ}}(k)$ for small k. For convenience, denote

$$P_k := (P_{\Delta^k,\Delta^k})_k : \mathbb{Z}(\Delta^k \times \Delta^k)_k \to \mathbb{Z}(\Delta^k \times \Delta^k)_{k+1}.$$

The proof of Theorem 4.3 tells us that $\delta_{\mathrm{EZ}}(k)$ is exactly the diameter of the chain $P_k([0,\dots,k]\times[0,\dots,k])$, where $P_k([0,\dots,k]\times[0,\dots,k])$ is defined inductively as follows: assuming that $P_{k-1}([0,\dots,k-1]\times[0,\dots,k-1])$ has been defined, P_{k-1} is determined by naturality and $P_k([0,\dots,k]\times[0,\dots,k]) \in \mathbb{Z}(\Delta^k \times \Delta^k)_{k+1}$ is defined to be a solution x of the system of linear equations

$$\partial x = (-P_{k-1}\partial + \nabla \circ \Delta - \mathrm{id})([0,\dots,k]\times[0,\dots,k]) \tag{4.3}$$

where $\partial : \mathbb{Z}(\Delta^k \times \Delta^k)_{k+1} \to \mathbb{Z}(\Delta^k \times \Delta^k)_k$ is viewed as a linear map.

We remark that

$$\operatorname{rank} \mathbb{Z}(\Delta^k \times \Delta^k)_{k+1} = \binom{2k+2}{k} \quad \text{and} \quad \operatorname{rank} \mathbb{Z}(\Delta^k \times \Delta^k)_k = \binom{2k+1}{k};$$

that is, the system (4.3) consists of $\binom{2k+1}{k}$ linear equations in $\binom{2k+2}{k}$ variables. It can be seen that the ranks grow exponentially by using Stirling's formula. Fortunately, for small k we can still find (or at least verify) solutions. We describe details below.

For $k=0$, $P_0([0]\times[0]) = 0$ satisfies (4.3) since $\nabla \circ \Delta = \mathrm{id}$ on $\mathbb{Z}(\Delta^0 \times \Delta^0)_0$. From this it follows that $\delta_{\mathrm{EZ}}(0) = 0$.

For $k=1$, straightforward computation shows that

$$\begin{aligned} \nabla\Delta([0,1]\times[0,1]) &= \nabla([0]\otimes[0,1] + [0,1]\otimes[1]) \\ &= [0,0]\times[0,1] + [0,1]\times[1,1]. \end{aligned}$$

Since it is equal to $\partial([0,0,1]\times[0,1,1])$, we have that

$$P_1([0,1]\times[0,1]) := [0,0,1]\times[0,1,1]$$

is a solution of (4.3). Since this is a chain of diameter 1, we have $\delta_{\mathrm{EZ}}(1) = 1$.

For $k = 2$, we have that

$$\begin{aligned}\nabla\Delta([0,1,2]\times[0,1,2]) &= \nabla([0]\otimes[0,1,2]+[0,1]\otimes[1,2]+[0,1,2]\otimes[2])\\ &= [0,0,0]\times[0,1,2]-[0,0,1]\times[1,2,2]\\ &\quad+[0,1,1]\times[1,1,2]+[0,1,2]\times[2,2,2]\end{aligned}$$

and that

$$\begin{aligned}&P_1\partial([0,1,2]\times[0,1,2])\\ &\quad= P_1([1,2]\times[1,2]-[0,2]\times[0,2]+[0,1]\times[0,1])\\ &\quad= [1,1,2]\times[1,2,2]-[0,0,2]\times[0,2,2]+[0,0,1]\times[0,1,1].\end{aligned}$$

By using these, it is straightforward to verify that

$$\begin{aligned}P_2([0,1,2]\times[0,1,2]) &= -[0,0,0,1]\times[0,1,2,2]+[0,0,1,1]\times[0,1,1,2]\\ &\quad+[0,0,1,2]\times[0,2,2,2]-[0,1,1,2]\times[0,1,2,2]\end{aligned}$$

is a solution of (4.3). Since its diameter is 4, we have $\delta_{\mathrm{EZ}}(2) = 4$.

For $k = 3$, (4.3) is a system of 1225 linear equations in 3136 variables. Aided by a computer, we found the following solution of (4.3):

$$\begin{aligned}&P_3([0,1,2,3]\times[0,1,2,3])\\ &\quad= [0,0,0,0,1]\times[0,1,2,3,3]-[0,0,0,1,1]\times[0,1,2,2,3]\\ &\qquad+[0,0,0,1,2]\times[0,2,3,3,3]+[0,0,1,1,1]\times[0,1,1,2,3]\\ &\qquad-[0,0,1,1,2]\times[0,2,2,3,3]+[0,0,1,2,2]\times[0,2,2,2,3]\\ &\qquad+[0,0,1,2,3]\times[0,3,3,3,3]+[0,1,1,1,2]\times[0,1,2,3,3]\\ &\qquad-[0,1,1,2,2]\times[0,1,2,2,3]-[0,1,1,2,3]\times[0,1,3,3,3]\\ &\qquad+[0,1,2,2,3]\times[0,1,2,3,3].\end{aligned}$$

We remark that we can verify by hand that it is a solution of (4.3). From this it follows that $\delta_{\mathrm{EZ}}(3) = d(P_3([0,1,2,3]\times[0,1,2,3])) = 11$.

For $k = 4$, our computation fully depends on a computer. A solution of the system (4.3), which has 15 876 equations in 44 100 variables in this case, is given by

$$\begin{aligned}&P_4([0,1,2,3,4]\times[0,1,2,3,4])\\ &\quad= -[0,0,0,0,0,1]\times[0,1,2,3,4,4]+[0,0,0,0,1,1]\times[0,1,2,3,3,4]\\ &\qquad+[0,0,0,0,1,2]\times[0,2,3,4,4,4]-[0,0,0,1,1,1]\times[0,1,2,2,3,4]\\ &\qquad-[0,0,0,1,1,2]\times[0,2,3,3,4,4]+[0,0,0,1,2,2]\times[0,2,3,3,3,4]\\ &\qquad-[0,0,0,1,2,3]\times[0,3,4,4,4,4]+[0,0,1,1,1,1]\times[0,1,1,2,3,4]\\ &\qquad+[0,0,1,1,1,2]\times[0,2,2,3,4,4]-[0,0,1,1,2,2]\times[0,2,2,3,3,4]\\ &\qquad+[0,0,1,1,2,3]\times[0,3,3,4,4,4]+[0,0,1,2,2,2]\times[0,2,2,2,3,4]\\ &\qquad-[0,0,1,2,2,3]\times[0,3,3,3,4,4]+[0,0,1,2,3,3]\times[0,3,3,3,3,4]+\end{aligned}$$

$$\begin{aligned}
&+ [0,0,1,2,3,4] \times [0,4,4,4,4,4] - [0,1,1,1,1,2] \times [0,1,2,3,4,4]\\
&+ [0,1,1,1,2,2] \times [0,1,2,3,3,4] - [0,1,1,1,2,3] \times [0,1,3,4,4,4]\\
&- [0,1,1,2,2,2] \times [0,1,2,2,3,4] + [0,1,1,2,2,3] \times [0,1,3,3,4,4]\\
&- [0,1,1,2,3,3] \times [0,1,3,3,3,4] - [0,1,1,2,3,4] \times [0,1,4,4,4,4]\\
&- [0,1,2,2,2,3] \times [0,1,2,3,4,4] + [0,1,2,2,3,3] \times [0,1,2,3,3,4]\\
&+ [0,1,2,2,3,4] \times [0,1,2,4,4,4] - [0,1,2,3,3,4] \times [0,1,2,3,4,4].
\end{aligned}$$

It follows that $\delta_{\mathrm{EZ}}(4) = 26$. □

Remark 4.6. In spite of Remark 4.5(2), it would be nicer if we had an explicit closed formula for $P_k([0,\dots,k] \times [0,\dots,k])$ for general k; this would give a general formula for the chain homotopy $P_{X,Y}$ for any X, Y and possibly a closed formula for $\delta_{\mathrm{EZ}}(k)$. The author does not know the answer.

4.3 Conjugation on Groups

Recall that for a group G, the (unnormalized) Moore complex $\mathbb{Z}BG_*$ associated to the simplicial classifying space BG (which is a simplicial set) can be used to compute the group homology $H_*(G)$ with integral coefficients. For example, see Appendix A.2 and A.4). In fact, $\mathbb{Z}BG_*$ is equal to the unnormalized bar resolution tensored with $\mathbb{Z}$. An explicit description of $\mathbb{Z}BG_*$ is as follows: $\mathbb{Z}BG_n$ is the free abelian group generated by $BG_n := \{[g_1,\dots,g_n] \mid g_i \in G\}$, and the boundary map $\partial : \mathbb{Z}BG_n \to \mathbb{Z}BG_{n-1}$ is given by $\partial c = \sum_{i=0}^{n} (-1)^i d_i c$, where d_i is defined by

$$d_i[g_1,\dots,g_n] = \begin{cases} [g_2,\dots,g_n] & \text{if } i = 0,\\ [g_1,\dots,g_{i-1}, g_i g_{i+1}, g_{i+2},\dots,g_n] & \text{if } 0 < i < n,\\ [g_1,\dots,g_{n-1}] & \text{if } i = n. \end{cases}$$

As abuse of notation, for a group homomorphism f, we denote by f the induced based chain map on $\mathbb{Z}B(-)_*$, that is, $f[g_1,\dots,g_n] = [f(g_1),\dots,f(g_n)]$.

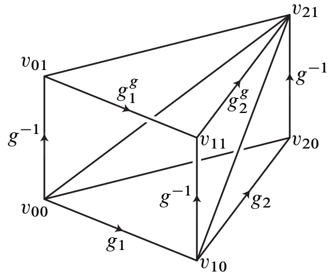
It is well known that for any group G and $g \in G$, the conjugation homomorphism $\mu_g : G \to G$ defined by $\mu_g(h) = h^g := ghg^{-1}$ induces the identity map on $H_*(G)$. For example, see [49, p. 191, theorem 6.7.8]. In the following theorem, we give a chain level statement in terms of controlled chain homotopies, from which the homological statement is immediately obtained.

THEOREM 4.7. *There is a family of chain homotopies*

$$\{S_{G,g} : \mathrm{id}_{\mathbb{Z}BG_*} \simeq \mu_g \mid G \text{ is a group}, g \in G\},$$

which is uniformly controlled by the function $\delta_{\mathrm{conj}}(k) := k + 1$. The chain homotopy $S_{G,g}$ is natural with respect to (G, g) in the sense that $fS_{G,g} = S_{\Gamma,f(g)}f$ for any homomorphism $f : G \to \Gamma$.

To motivate our chain homotopy construction for Theorem 4.7, we recall a geometric interpretation of an n-simplex $[g_1,\dots,g_n]$ of BG that arises from the nerve



$$[v_{00}, v_{10}, v_{20}, v_{21}] \mapsto [g_1, g_2, g^{-1}]$$
$$[v_{00}, v_{10}, v_{11}, v_{21}] \mapsto [g_1, g^{-1}, g_2^g]$$
$$[v_{00}, v_{01}, v_{11}, v_{21}] \mapsto [g^{-1}, g_1^g, g_2^g]$$

FIGURE 4.1. Prism decomposition of a homotopy for conjugation.

construction for G: there is exactly one 0-simplex $[\,]$ in BG that is the base point, and for $n > 0$, $[g_1, \dots, g_n] \in BG_n$ corresponds to an n-simplex $[v_0, \dots, v_n]$ (which is possibly degenerate) in the geometric realization of BG whose edge $[v_{i-1}, v_i]$ is a loop representing $g_i \in \pi_1(BG) = G$.

Consider a prism $\Delta^n \times [0,1]$. For convenience, we write $\Delta^n = [e_0 \dots, e_n]$ and denote the vertices of $\Delta^n \times [0,1]$ by $v_{ij} = (e_i, j)$, $i = 0, \dots, n$, $j = 0, 1$. If there is a geometric homotopy from id_{BG} to the conjugation μ_g, then the restriction on a simplex $[g_1, \dots, g_n]$ should give a map of $\Delta^n \times [0,1]$ that sends the edges $[v_{(i-1)0}, v_{i0}]$ and $[v_{(i-1)1}, v_{i1}]$ to g_i and $\mu_g(g_i) = g_i^g$, respectively. This tells us what the restriction $\Delta^n \times \{0,1\} \to BG$ should be. The standard prism decomposition divides the product $\Delta^n \times [0,1]$ into $n+1$ simplices. It turns out that, for instance as illustrated in Figure 4.1 for $n = 2$, we can label edges of the resulting simplices in such a way that the prescribed $\Delta^n \times \{0,1\} \to BG$ extends to $\Delta^n \times [0,1]$ simplicially. Note that in Figure 4.1 each path $e_i \times [0,1]$ is sent to the loop g^{-1}, so that the basepoint change effect of the homotopy is exactly the conjugation by g on $\pi_1(BG) = G$.

Generalizing Figure 4.1 to an arbitrary dimension n, we obtain the chain homotopy formula used in the formal proof of Theorem 4.7 given below.

PROOF OF THEOREM 4.7. For a group G and an element $g \in G$, we define a chain homotopy
$$S = S_{G,g} : \mathbb{Z}BG_* \to \mathbb{Z}BG_{*+1}$$
by
$$S[g_1, \dots, g_n] = \sum_{i=0}^{n} (-1)^i \big[g_1, \dots, g_i, g^{-1}, g_{i+1}^g, \dots, g_n^g\big].$$
By a straightforward computation it is verified that $S\partial + \partial S = \mu_g - \mathrm{id}$. From the defining formula, it follows that $S_{G,g}$ is natural and that $d_{S_{G,g}}(k) \le k+1$. $\square$

5 Chain Homotopy for Embeddings into Mitoses

We begin by recalling a definition of Baumslag, Dyer, and Heller to set up notation. As before, we write $g^h := hgh^{-1}$.

DEFINITION 5.1 ([3]). Suppose G is a group. A group M endowed with an embedding $\imath : G \to M$ is a *mitosis* of G if there are elements $u, t \in M$ such that M is generated by $\imath(G) \cup \{u, t\}$ and $g^t = gg^u$, $[h, g^u] = e$ for any $g, h \in \imath(G)$. In particular, define

$$m(G) := \langle G, u, t \mid [h, g^u] = e,\ g^t = gg^u \text{ for any } g, h \in G\rangle.$$

Then $m(G)$ together with the natural embedding $k_G : G \to m(G)$ is a mitosis of G.

Define $\mathbb{A}^0(G) = G$, $\mathbb{A}^n(G) := m(\mathbb{A}^{n-1}(G))$ for $n \geq 1$ inductively. We denote by $i_G^n : G \to \mathbb{A}^n(G)$ the composition $k_{\mathbb{A}^{n-1}(G)} \circ \cdots \circ k_{\mathbb{A}^1(G)} \circ k_G$.

As observed in [3], it is verified straightforwardly that (i) $m : \mathbf{Gp} \to \mathbf{Gp}$ is a functor of the category $\mathbf{Gp}$ of groups, (ii) k_G is a natural transformation $\mathrm{id}_{\mathbf{Gp}} \to m$ that is injective for each G, and (iii) $m(f) : m(G) \to m(\Gamma)$ is injective whenever $f : G \to \Gamma$ is an injective group homomorphism. Consequently (i), (ii), and (iii) hold for $(\mathbb{A}^n, i_G^n)$ in place of (m, k_G).

In [3], they showed that if $\Bbbk$ is a field, then the map $H_i(G; \Bbbk) \to H_i(\mathbb{A}^n(G); \Bbbk)$ induced by i_G^n is 0 for $i = 1, \ldots, n$. Our main aim of this section is to prove the following chain level result (Theorem 5.2), which particularly gives this homological result of [3] as an immediate consequence.

We denote the trivial group homomorphism by $e_{\pi,G} : \pi \to G$. When the groups π and G are understood from the context, we write $e = e_{\pi,G}$ by dropping π, G from the notation. Recall that we denote by $f : \mathbb{Z}BG_* \to \mathbb{Z}B\Gamma_*$ the chain map induced by a group homomorphism $f : G \to \Gamma$.

THEOREM 5.2. *For each n, there is a family*

$$\left\{\Phi_G^n : e \simeq i_G^n \mid G \text{ is a group}\right\}$$

of partial chain homotopies Φ_G^n defined in dimension $\leq n$, between the chain maps $e, i_G^n : \mathbb{Z}BG_* \to \mathbb{Z}B\mathbb{A}^n(G)_*$, which is uniformly controlled by a function δ_{BDH}. For $k \leq 4$, the value of $\delta_{\mathrm{BDH}}(k)$ is as follows:

k	0	1	2	3	4
$\delta_{\mathrm{BDH}}(k)$	0	6	26	186	3410

A precise definition of δ_{BDH} will be given in Definition 5.7. Note that the control function δ_{BDH} is independent of n. The values of $\delta_{\mathrm{BDH}}(k)$ for $k \leq 3$ will be essential in proving Theorem 1.5 stated in the introduction.

The remainder of this section is devoted to the proof of Theorem 5.2. As a preliminary, we make some observations on the product of groups. From the definition, for groups G and H, we have $B(G \times H) = BG \times BH$ as simplicial sets. Let

$$\Delta = \Delta_{BG,BH} : \mathbb{Z}(BG \times BH)_* \to \mathbb{Z}BG_* \otimes \mathbb{Z}BH_*$$

be the Alexander-Whitney map. We define

$$\Lambda_G, \Lambda_H, \Lambda : \mathbb{Z}(BG \times BH)_* \to \mathbb{Z}BG_* \otimes \mathbb{Z}BH_*$$

by

$$\Lambda_G(\sigma \times \tau) := \sigma \otimes (d_0)^n \tau = \sigma \otimes [\,],$$
$$\Lambda_H(\sigma \times \tau) := d_1 \cdots d_n \sigma \otimes \tau = [\,] \otimes \tau,$$

for $\sigma \times \tau \in (BG \times BH)_n$, and by $\Lambda := \Delta - \Lambda_G - \Lambda_H$. Note that if $n \geq 1$, Λ_H and Λ_G are the first and last terms of the defining formula (4.1) of Δ, respectively. Consequently, Λ is the sum of the remaining terms.

LEMMA 5.3. *The maps Λ_G, Λ_H, and Λ are chain maps.*

PROOF. Since

$$\begin{aligned}\Lambda_H \partial(\sigma \times \tau) &= \Lambda_H\Big(\sum_i (-1)^i d_i\sigma \times d_i\tau\Big) \\ &= \sum_i (-1)^i ([\,] \otimes d_i\tau) = [\,] \otimes \partial\tau = \partial\Lambda_H(\sigma \times \tau),\end{aligned}$$

we have that Λ_H is a chain map. A similar argument works for Λ_G. Since Δ is a chain map, it follows that $\Lambda = \Delta - \Lambda_G - \Lambda_H$ is a chain map. □

For the next lemma, recall that $\delta_{\mathrm{EZ}}(k)$ is the control function in Theorem 4.4.

LEMMA 5.4. *Suppose $f : G \to K$ and $g : H \to L$ are group homomorphisms. Suppose $Q : e \simeq f$ is a partial chain homotopy defined in dimension $\leq n-1$ between $e, f : \mathbb{Z}BG_* \to \mathbb{Z}BK_*$, that is, $Q\partial + \partial Q = f - e$ on $\mathbb{Z}BG_i$ for $i \leq n-1$. Suppose $Q_0 = 0$ on $\mathbb{Z}BG_0$. Consider the product homomorphisms*

$$f \times g,\, f \times e,\, e \times g : G \times H \to K \times L$$

and the induced chain maps $\mathbb{Z}(BG \times BH)_* \to \mathbb{Z}(BK \times BL)_*$. Let $P = P_{BK,BL} : \nabla\Delta \simeq \mathrm{id}$ be the chain homotopy in Theorem 4.4. Then

$$T := P(f \times g - e \times g) + \nabla(Q \otimes g)\Lambda : \mathbb{Z}(BG \times BH)_* \to \mathbb{Z}(BK \times BL)_{*+1}$$

is a partial chain homotopy

$$T : (f \times e - e \times e) + (e \times g - e \times e) \simeq (f \times g - e \times e)$$

defined in dimension $\leq n$. Furthermore, it satisfies that $T_0 = 0$ on $C_0(BK \times BL)$, that is, $d_T(0) = 0$, and

$$d_T(k) \leq 2 \cdot \delta_{\mathrm{EZ}}(k) + (k-1)\binom{k}{\lfloor k/2 \rfloor} \cdot d_Q(k-1) \quad \text{for } k \geq 1.$$

We remark that Δ, Λ, and ∇ in the above statements are those for the product of BK and BL.

PROOF. By Lemma 4.1(3), we have that $Q \otimes g : e \otimes g \simeq f \otimes g$ is a partial chain homotopy. More precisely, on $\sum_{i<n} \mathbb{Z}BG_i \otimes \mathbb{Z}BH_*$,

$$\begin{aligned}(Q \otimes g)\partial + \partial(Q \otimes g) &= Q\partial \otimes g \pm Q \otimes g\partial + \partial Q \otimes g \mp Q \otimes \partial g \\ &= (Q\partial + \partial Q) \otimes g = f \otimes g - e \otimes g.\end{aligned} \tag{5.1}$$

By the definitions, for any f and g, the following diagram commutes:

$$\begin{tikzcd}
\mathbb{Z}(BG \times BH)_* \arrow[r, "f\times g"] \arrow[d, "\Delta"'] & \mathbb{Z}(BK \times BL)_* \arrow[d, "\Delta"] \\
\mathbb{Z}BG_* \otimes \mathbb{Z}BH_* \arrow[r, "f\otimes g"'] & \mathbb{Z}BK_* \otimes \mathbb{Z}BL_*
\end{tikzcd} \tag{5.2}$$

We also have

$$\begin{aligned}
\nabla(f \otimes g)\Lambda_G(\sigma \times \tau) &= \nabla(f \otimes g)(\sigma \otimes [\,]) \\
&= \nabla(f\sigma \otimes [\,]) = (f \times e)(\sigma \times \tau)
\end{aligned} \tag{5.3}$$

for any f and g. Similarly,

$$\nabla(f \otimes g)\Lambda_H = e \times g. \tag{5.4}$$

Now, on $\mathbb{Z}(BG \times BH)_k$ with $1 \le k \le n$, we have

$$\begin{aligned}
f \times g - e \times g &\simeq \nabla\Delta(f \times g - e \times g) && \text{by Theorem 4.4} \\
&= \nabla(f \otimes g - e \otimes g)\Delta && \text{by (5.2)} \\
&= \nabla(f \otimes g - e \otimes g)(\Lambda_G + \Lambda_H + \Lambda) && \text{by definitions} \\
&= (f \times e - e \times e) + (e \times g - e \times g) \\
&\quad + \nabla((Q \otimes g)\partial + \partial(Q \otimes g))\Lambda && \text{by (5.3), (5.4), (5.1)} \\
&= (f \times e - e \times e) \\
&\quad + \nabla(Q \otimes g)\Lambda\partial + \partial\nabla(Q \otimes g)\Lambda && \text{by Lemma 5.3.}
\end{aligned} \tag{5.5}$$

Note that in (5.5) we can apply (5.1) since the image of $\mathbb{Z}(BG \times BH)_k$ under Λ lies in $\sum_{i=1}^{k-1} \mathbb{Z}BG_i \otimes \mathbb{Z}BH_*$.

On $\mathbb{Z}(BG \times BH)_0$, we have $f \times g - e \times g = 0 = f \times e - e \times e$.

Let $P = P_{BK,BL}$ be the chain homotopy given by Theorem 4.4, and let

$$T := P(f \times g - e \times g) + \nabla(Q \otimes g)\Lambda.$$

Note that $T_0 = 0$ on $\mathbb{Z}(BG \times BH)_0$ since $Q_0 = 0$. From (5.5) and Lemma 4.1(1) and (2), it follows that T is a partial chain homotopy between $(f \times e - e \times e) + (e \times g - e \times e)$ and $f \times g - e \times e$ in dimension $\le n$.

Now we estimate the diameter $d_T(k)$ of T. The chain maps $f \times g$ and $e \times g$ have diameter function $\equiv 1$. Observe that the defining formula (4.2) for ∇ has $\binom{p+q}{p}$ summands, since the number of (p,q)-shuffles is $\binom{p+q}{p}$. It follows that $d_\nabla(k) \le \binom{k}{\lfloor k/2 \rfloor}$. Similarly, from the defining formula (4.1) for Δ, it follows that $d_\Lambda(k) \le k - 1$. Note that $d_{(Q\otimes g)\Lambda}(k) \le d_Q(k-1) \cdot d_\Lambda(k)$ since the Q factor in the expression $(Q \otimes g)\Lambda$ is applied only to chains of dimension at most $k-1$. Combining the above observations using Lemma 4.1, we obtain the claimed estimate for $d_T(k)$. $\square$

Remark 5.5. A *reduced simplicial set* is defined to be a simplicial set with a unique 0-simplex. Lemmas 5.3 and 5.4 hold for reduced simplicial sets, although we stated and proved them for classifying spaces of groups only. The proofs are identical.

We use the above results to show a key property of the mitosis embedding $k_G : G \to m(G)$ on the chain level.

THEOREM 5.6. *Suppose $\phi : \pi \to G$ is a group homomorphism and $Q : e \simeq \phi$ is a partial chain homotopy defined in dimension $\le n-1$ between $e, \phi : \mathbb{Z}B\pi_* \to \mathbb{Z}BG_*$ such that $Q_0 = 0$ on $\mathbb{Z}B\pi_0$. Then there is a partial chain homotopy $R : e \simeq k_G \circ \phi$ defined in dimension $\le n$ between $e, k_G \circ \phi : \mathbb{Z}B\pi_* \to \mathbb{Z}Bm(G)_*$. In addition, $R_0 = 0$ on $\mathbb{Z}B\pi_0$; that is, $d_R(0) = 0$ and*

$$d_R(k) \le 2(k+1) + 2\cdot\delta_{\mathrm{EZ}}(k) + (k-1)\binom{k}{\lfloor k/2\rfloor}\cdot d_Q(k-1) \quad \text{for } k \ge 1.$$

PROOF. Recall that

$$m(G) = \langle G, u, t \mid [h, g^u] = e,\ g^t = gg^u \text{ for any } g, h \in G\rangle.$$

Define inclusions $i, j, D : \pi \to \pi \times \pi$ by $i(g) = (g, e)$, $j(g) = (e, g)$, and $D(g) = (g, g)$. Define $f : G \times G \to m(G)$ by $f(g, h) = gh^u$. Recall $\mu_g(h) = h^g$ denotes the conjugation by g. Consider the following diagram:

$$\begin{tikzcd}
 & \mathbb{Z}(B\pi \times B\pi)_* \arrow[r, "\phi\times\phi"] & \mathbb{Z}(BG \times BG)_* \arrow[r, "f"] & \mathbb{Z}Bm(G)_* \\
\mathbb{Z}B\pi_* \arrow[ur, "j"] \arrow[r, "i"] \arrow[dr, "D"'] & \mathbb{Z}(B\pi \times B\pi)_* \arrow[r, "\phi\times\phi"] & \mathbb{Z}(BG \times BG)_* \arrow[r, "f"] & \mathbb{Z}Bm(G)_* \arrow[u, "\mu_u"'] \arrow[d, "\mu_t"] \\
 & \mathbb{Z}(B\pi \times B\pi)_* \arrow[r, "\phi\times\phi"] & \mathbb{Z}(BG \times BG)_* \arrow[r, "f"] & \mathbb{Z}Bm(G)_*
\end{tikzcd}$$

It commutes since it is obtained from a commutative diagram of group homomorphisms.

For $g \in m(G)$, let $S_g := S_{m(G),g} : \mathrm{id} \simeq \mu_g$ be the chain homotopy in Theorem 4.7. Then we obtain a chain homotopy

$$S_u f(\phi\times\phi)i : f(\phi\times\phi)i \simeq \mu_u f(\phi\times\phi)i = f(\phi\times\phi)j \tag{5.6}$$

by Lemma 4.1(2). Similarly, we obtain a chain homotopy

$$S_t f(\phi\times\phi)i : f(\phi\times\phi)i \simeq f(\phi\times\phi)D. \tag{5.7}$$

Since $Q : e \simeq \phi$, Lemma 5.4 gives us a partial chain homotopy,

$$T : (\phi\times e - e\times e) + (e\times\phi - e\times e) \simeq \phi\times\phi - e\times e$$

in dimension $\le n$. From this we obtain a partial chain homotopy

$$fTD : f(\phi\times e + e\times\phi - e\times e)D \simeq f(\phi\times\phi)D$$

in dimension $\le n$, by Lemma 4.1(2). Since

$$f(\phi\times e)D = f(\phi\times\phi)i, \quad f(e\times\phi)D = f(\phi\times\phi)j, \quad f(e\times e)D = e,$$

it follows that fTD is indeed a chain homotopy

$$fTD : f(\phi\times\phi)i + f(\phi\times\phi)j - e \simeq f(\phi\times\phi)D. \tag{5.8}$$

Now we have

$$\begin{aligned} k_G \circ \phi - e = f(\phi \times \phi)i - e &\simeq f(\phi \times \phi)D - f(\phi \times \phi)j && \text{by (5.8)} \\ &\simeq f(\phi \times \phi)i - f(\phi \times \phi)j && \text{by (5.7)} \\ &\simeq f(\phi \times \phi)j - f(\phi \times \phi)j = 0 && \text{by (5.6).} \end{aligned}$$

Also, Lemma 4.1(1) tells us that

$$R := fTD - S^t f(\phi \times \phi)i + S^u f(\phi \times \phi)i$$

is a chain homotopy $R : e \simeq k_G \circ \phi$. Since $Q_0 = 0$ by the hypothesis, we have $T_0 = 0$ by Lemma 5.4. From this it follows that $R_0 = 0$, that is, $d_R(0) = 0$. Also, by Lemma 4.1 (1) and by the estimates in Theorem 4.7 and Lemma 5.4, we obtain

$$\begin{aligned} d_R(k) &\le d_{S^t}(k) + d_{S^u}(k) + d_T(k) \\ &\le 2(k+1) + 2 \cdot \delta_{\mathrm{EZ}}(k) + (k-1)\binom{k}{\lfloor k/2 \rfloor} \cdot d_Q(k-1) \quad \text{for } k \ge 1. \; \square \end{aligned}$$

Applying Theorem 5.6 repeatedly, we obtain the following result for $i_G^n : G \to \mathbb{A}^n(G)$. For the statement, we need a definition.

DEFINITION 5.7. Let $\delta_{\mathrm{BDH}} : \{0, \ldots, n\} \to \mathbb{Z}_{\ge 0}$ be the function defined inductively by the initial condition $\delta_{\mathrm{BDH}}(0) = 0$ and the recurrence relation

$$\delta_{\mathrm{BDH}}(k) = 2(k+1) + 2 \cdot \delta_{\mathrm{EZ}}(k) + (k-1)\binom{k}{\lfloor k/2 \rfloor} \cdot \delta_{\mathrm{BDH}}(k-1)$$

for $k \ge 1$.

COROLLARY 5.8. *For each integer $n \ge 0$, there is a family*

$$\{\Phi_G^n : e \simeq i_G^n \mid G \text{ is a group}\}$$

of partial chain homotopies in dimension $\le n$ between $e, i_G^n : \mathbb{Z}BG_* \to \mathbb{Z}B\mathbb{A}^n(G)_*$, which is uniformly controlled by δ_{BDH}.

PROOF. For $n = 0$, the zero map $\Phi_G := 0$ is a partial chain homotopy $\Phi^G : e \simeq \mathrm{id}_G = i_G^0$ in dimension ≤ 0. So the claimed conclusion holds.

Suppose the conclusion holds for $n-1$. Applying Theorem 5.6 to $\phi := i_G^{n-1} : G \to \mathbb{A}^{n-1}(G)$ and $Q := \Phi_G^{n-1} : e \simeq i_G^{n-1}$, it follows that there is a partial chain homotopy

$$\Phi_G^n : e \simeq k_{\mathbb{A}^{n-1}G} \circ i_G^{n-1} = i_G^n$$

in dimension $\le n$ that satisfies $d_{\Phi_G^n}(0) = 0$ and

$$d_{\Phi_G^n}(k) \le 2(k+1) + 2 \cdot \delta_{\mathrm{EZ}}(k) + (k-1)\binom{k}{\lfloor k/2 \rfloor} \cdot d_{\Phi_G^{n-1}}(k-1) \quad \text{for } k \ge 1.$$

Since $\{\Phi_G^{n-1}\}$ is uniformly controlled by δ_{BDH}, the conclusion for n follows. $\square$

Now we are ready to complete the proof of Theorem 5.2 stated in the beginning of this section.

PROOF OF THEOREM 5.2. The existence of the desired uniformly controlled family of chain homotopies in Theorem 5.2 is no more than Corollary 5.8. For $k \le 4$, the values of $\delta_{\mathrm{BDH}}(k)$ are obtained by an inductive straightforward computation by using Definition 5.7 and the values of $\delta_{\mathrm{EZ}}(k)$ given in Theorem 4.4. $\square$

6 Explicit Universal Bounds from Presentations of 3-Manifolds

In this section we obtain explicit estimates of the Cheeger-Gromov universal bound from fundamental presentations of 3-manifolds.

6.1 Bounds from Triangulations

The goal of this subsection is to give a proof of Theorem 1.5: *suppose M is a 3-manifold with simplicial complexity n. Then for any $\phi : \pi_1(M) \to G$,*

$$|\rho^{(2)}(M, \phi)| \le 363\,090 \cdot n.$$

Recall that the *simplicial complexity* of a 3-manifold M is the minimal number of 3-simplices in a triangulation (i.e., a simplicial complex structure) of M.

In the proof, we will use the results developed in Sections 3, 4, and 5, as well as the idea of the existence proof of Theorem 1.3 given in Section 2. First we state a corollary of Theorem 3.9 and Corollary 5.8. Recall that we defined the functorial embedding $i^n_G : G \to \mathbb{A}^n(G)$ in Definition 5.1.

THEOREM 6.1. *Suppose M is a 3-manifold with simplicial complexity n. View M as a manifold over $\mathbb{A}^3(\pi_1(M))$ via the embedding $i^3_{\pi_1(M)} : \pi_1(M) \to \mathbb{A}^3(\pi_1(M))$. Then there is a smooth bordism W over $\mathbb{A}^3(\pi_1(M))$ between M and a trivial end whose 2-handle complexity is at most $181\,545 \cdot d(\zeta_M)$.*

In the proof of Theorem 6.1 given below, there is a small technicality that arises from the fact that we use two chain complexes for a simplicial set X: the cellular chain complex $C_*(X)$ of its geometric realization, which was used in Section 3, and the Moore complex $\mathbb{Z}X_*$ of the simplicial abelian group $\mathbb{Z}X$ associated to X, which was used in Sections 4 and 5. It is known that if we denote by $D_*(X)$ the subgroup of $\mathbb{Z}X_*$ generated by degenerate simplices of X, then $D_*(X)$ is indeed a subcomplex, $C_*(X) \cong \mathbb{Z}X_*/D_*(X)$, and the projection $p : \mathbb{Z}X_* \to C_*(X)$ is a chain homotopy equivalence [38, p. 236]. See Appendix A.2 for more details.

PROOF OF THEOREM 6.1. We write $\pi := \pi_1(M)$, $\Gamma := \mathbb{A}^3(\pi_1(M))$, and $i := i^3_{\pi_1(M)} : \pi \to \Gamma$ for brevity. Choose a simplicial complex structure of M with a minimal number of 3-simplices. By abuse of notation, we denote by M the simplicial set obtained from this simplicial complex structure. As before, let $\zeta_M \in C_*(M)$ be the sum of oriented 3-simplices of M that represents the fundamental class $[M] \in H_3(M)$. Since M is a simplicial complex, $C_*(X)$ is a subcomplex of $\mathbb{Z}_*X$, and the projection $p : \mathbb{Z}X_* \to C_*(X)$ is a left inverse of the inclusion. In particular, ζ_M lifts to a cycle $\xi_M \in \mathbb{Z}M_3$. We have $d(\xi_M) = d(\zeta_M)$.

By Theorem 3.7 (see also Proposition A.1 in the appendix), the identity map $\pi_1(M) \to \pi = \pi_1(B\pi)$ induces a simplicial-cellular map $j : M \to B\pi$. Let $\phi = i \circ j : M \to B\pi \to B\Gamma$. By Theorem 5.2, there is a partial chain homotopy $\Phi : e \simeq i$ defined in dimension ≤ 3. (Using our convention, here e and i designate the induced chain maps $\mathbb{Z}B\pi_* \to \mathbb{Z}B\Gamma_*$.) Since ξ_M is a cycle, we have

$$\begin{aligned}\phi(\xi_M) = i(j(\xi_M)) &= e(j(\xi_M)) + \Phi\partial(j(\xi_M)) + \partial\Phi(j(\xi_M)) \\ &= e(j(\xi_M)) + \partial\Phi(j(\xi_M))\end{aligned} \tag{6.1}$$

in $\mathbb{Z}B\Gamma_3$. Note that the image of $e : \mathbb{Z}B\Gamma_i \to \mathbb{Z}B\Gamma_i$ lies in $D_i(B\Gamma)$ for $i > 0$. By applying the projection $p : \mathbb{Z}B\Gamma_* \to C_*(B\Gamma)$ to (6.1), it follows that the 4-chain $u := p\Phi(j(\xi_M))$ satisfies $\phi_\#(\zeta_M) = \partial u$ in the cellular chain complex $C_*(B\Gamma)$. Here we use that $p\phi = \phi_\# p$ for a morphism ϕ of simplicial sets.

Theorem 5.2 also tells us that $d_\Phi(3) \le \delta_{\mathrm{BDH}}(3) = 186$. We have $d_j(k) = d_p(k) = 1$ since j is (induced by) a simplicial map and p is a projection sending a basis element to a basis element or 0. From this it follows that

$$d(u) = d\big(p\big(\Phi(j(\xi_M))\big)\big) \le d_p(3) \cdot d_\Phi(3) \cdot d_j(3) \cdot d(\xi_M) = 186 \cdot d(\zeta_M).$$

Now we apply Theorem 3.9 to (M, ϕ, u). This gives us a smooth bordism W over Γ between M and another 3-manifold N that is trivially over $B\Gamma$, where

$$(\text{2-handle complexity of } W) \le 195 \cdot d(\zeta_M) + 975 \cdot d(u) \le 181\,545 \cdot d(\zeta_M). \quad \square$$

PROOF OF THEOREM 1.5. Suppose M is a closed 3-manifold with simplicial complexity n, and $\phi : \pi_1(M) \to G$ is a homomorphism. By Theorem 6.1, there is a smooth bordism W with $\partial W = M \sqcup -N$ over $\mathbb{A}^3(\pi_1(M))$, where N is trivially over $\mathbb{A}^3(\pi_1(M))$ and the 2-handle complexity of W is at most $181\,545 \cdot n$. Let $\Gamma := \mathbb{A}^3(G)$. Similarly to the proof of Theorem 1.3, we consider the following commutative diagram:

$$\begin{tikzcd}
\pi_1(M) \arrow[rr, "\phi"] \arrow[dr, hook, "i^3_{\pi_1(M)}"] \arrow[dd] & & G \arrow[dr, hook, "i^3_G"] & \\
& \mathbb{A}^3(\pi_1(M)) \arrow[rr, "\mathbb{A}(\phi)"] & & \mathbb{A}^3(G) = \Gamma \\
\pi_1(W) \arrow[ur] & & &
\end{tikzcd}$$

By L^2-induction and Remark 3.3, we can compute the ρ-invariant as the L^2-signature defect of W as follows:

$$\rho^{(2)}(M, \phi) = \rho^{(2)}\big(M, i^3_G \circ \phi\big) = \operatorname{sign}^{(2)}_\Gamma W - \operatorname{sign} W.$$

Since both $|\operatorname{sign}^{(2)}_\Gamma W|$ and $|\operatorname{sign} W|$ are not greater than the 2-handle complexity of W, it follows that

$$|\rho^{(2)}(M,\phi)| \le 2 \cdot 181\,545 \cdot n = 363\,090 \cdot n. \qquad \square$$

6.2 Bounds from Heegaard Splittings and Surgery Presentations

In this subsection, we first prove Theorem 1.8, which says the following: *if M is a closed 3-manifold with Heegaard-Lickorish complexity ℓ, then for any ϕ,*

$$|\rho^{(2)}(M,\phi)| \le 251\,258\,280 \cdot \ell.$$

Our proof relies on Theorem 1.5 and a result from [6]:

THEOREM 6.2 ([6, theorem A]). *Suppose M is a closed 3-manifold with simplicial complexity n and Heegaard-Lickorish complexity ℓ. If $M \ne \mathbb{S}^3$, then $n \le 692\ell$.*

PROOF OF THEOREM 1.8. If $M = \mathbb{S}^3$, then since M is simply connected, $\rho^{(2)}(\mathbb{S}^3,\phi) = 0$ for any ϕ. It follows that the conclusion holds in this case. Suppose $M \ne \mathbb{S}^3$ has Heegaard-Lickorish complexity ℓ. Then by Theorems 6.2 and 1.5, it follows that

$$|\rho^{(2)}(M,\phi)| \le 363\,090 \cdot 692 \cdot \ell = 251\,258\,280 \cdot \ell$$

for any ϕ. $\square$

In the rest of this subsection, we prove Theorem 1.9. Recall that $c(L)$ denotes the crossing number of a link L. Also recall that for a framed link L, we define $f(L) = \sum_i |n_i|$ where $n_i \in \mathbb{Z}$ is the framing on the i^{th} component of L. Theorem 1.9 says the following: *suppose M is a 3-manifold obtained by surgery along a framed link L in $\mathbb{S}^3$. Then for any ϕ,*

$$|\rho^{(2)}(M,\phi)| \le 69\,713\,280 \cdot c(L) + 34\,856\,640 \cdot f(L).$$

For the proof of Theorem 1.9, we need the following result proven in [6].

THEOREM 6.3 ([6, theorem B and def. 1.3]). *Suppose $M \ne \mathbb{S}^3$ is a 3-manifold obtained by surgery along a framed link L in $\mathbb{S}^3$ that has no split unknotted zero framed component. Then the simplicial complexity of M is not greater than $192 \cdot c(L) + 96 \cdot f(L)$.*

PROOF OF THEOREM 1.9. If M is $\mathbb{S}^3$, then $\rho^{(2)}(M,\phi) = 0$ for any ϕ. Therefore we may assume that $M \ne \mathbb{S}^3$.

Suppose L is a framed link in $\mathbb{S}^3$ that gives M by surgery. We claim that we may assume that L does not have any split unknotted zero framed component. To show the claim, suppose L has k split unknotted zero framed components, and let L' be the sublink consisting of the other components. Let M and M' be the 3-manifolds obtained by surgery on L and L', respectively. Then M is the connected sum of M' and k copies of $\mathbb{S}^1 \times \mathbb{S}^2$. Since $\mathbb{S}^1 \times \mathbb{S}^2 = \partial(\mathbb{S}^1 \times D^3)$ over $\pi_1(\mathbb{S}^1 \times \mathbb{S}^2) = \mathbb{Z}$ and $\mathbb{S}^1 \times D^3$ has no 2-handles, $\rho^{(2)}(\mathbb{S}^1 \times \mathbb{S}^2, \psi) = 0$ for any ψ. Since $\rho^{(2)}$ is additive under a connected sum, we have $\rho^{(2)}(M,\phi) = \rho^{(2)}(M',\phi')$ where

$\phi' : \pi_1(M') \to G$ is the homomorphism induced by $\phi : \pi_1(M) \to G$. Since $c(L) = c(L')$, $f(L) = f(L')$, and since we are interested in a universal bound, it follows that we may assume $L = L'$ as claimed.

By the claim and by Theorem 6.3, the simplicial complexity of M is at most $192 \cdot c(L) + 96 \cdot f(L)$. By Theorem 1.5, it follows that

$$\begin{aligned} |\rho^{(2)}(M,\phi)| &\le 363\,090(192 \cdot c(L) + 96 \cdot f(L)) \\ &= 69\,713\,280 \cdot c(L) + 34\,856\,640 \cdot f(L) \end{aligned}$$

for any homomorphism $\phi : \pi_1(M) \to G$. □

The following theorem gives a similar but better estimate for a special case:

THEOREM 6.4. *Suppose D is a planar diagram of a link L with c crossings in which each component is involved in a crossing. Let M be the 3-manifold obtained by surgery on L along the blackboard framing of D. Then*

$$|\rho^{(2)}(M,\phi)| \le 34\,856\,640 \cdot c$$

for any homomorphism $\phi : \pi_1(M) \to G$.

PROOF. We proceed similarly to the proof of Theorem 1.9; instead of Theorem 6.3, we apply [6, lemma 2.1] to our case to obtain that the simplicial complexity of M is at most $96c$. The conclusion follows from Theorem 1.5. □

EXAMPLE 6.5. Consider the stevedore knot, which is 6_1 in the table in Rolfsen [47] or KnotInfo [9]. It is the simplest nontrivial ribbon knot. It has a 6-crossing diagram with 2 crossings of the same sign and 4 crossings of the opposite sign. By applying the Reidemeister move I twice, we obtain an 8-crossing diagram with writhe zero. Since the blackboard framing is the zero framing for this diagram, it follows that the zero surgery manifold M of 6_1 satisfies $|\rho^{(2)}(M,\phi)| \le 34\,856\,640 \cdot 8 = 278\,853\,120$ for any ϕ, by Theorem 6.4.

Remark 6.6. In light of Theorem 1.9 and Theorem 6.4, now the proofs of the following existence results of various authors can give us explicit examples of the following:

(1) knots of infinite order in the graded quotient of the Cochran-Orr-Teichner n-solvable filtration, and similarly for the grope filtration [19, theorems 1.4 and 4.2], [15, theorems 9.1 and 9.5, cor. 9.7];
(2) slice knots that are algebraically doubly slice but nontrivial in the graded quotient of the double n-solvable filtration (and consequently not doubly slice) [32, theorem 1.1];
(3) knots whose iterated Bing doubles are n-solvable but not $(n+1)$-solvable (and consequently not slice) [14, cor. 5.2 and 5.3, theorem 5.16];
(4) 2-torsion knots generating $(\mathbb{Z}_2)^\infty$ in the graded quotients of the n-solvable filtration [16, theorems 5.5 and 5.7, cor. 5.6];
(5) nonconcordant knots obtained from the same knots by infection using distinct curves [22, theorem 3.1 and cor. 3.2 and 3.3];

(6) knots that generate $\mathbb{Z}^\infty$ in the graded quotients of the n-solvable filtration and have vanishing Cochran-Orr-Teichner PTFA signature obstructions [4, theorems 1.4 and 4.11];
(7) links that are height n grope concordant to but not height $n.5$ Whitney tower concordant to the Hopf link [5, theorem 4.1];
(8) nonconcordant m-component links with the same arbitrarily given multivariable Alexander polynomial Δ, if $m > 2$ or $\Delta \neq 1$ [8, theorems A, B, 3.1, and 4.1];
(9) nonconcordant links admitting a homology cobordism between their zero surgery manifolds in which the meridians are homotopic [11, theorems 1.1 and 1.2].

7 Complexity of 3-Manifolds

In this section, we present applications of our Cheeger-Gromov bounds to the complexity of 3-manifolds. We will also show that the Cheeger-Gromov bounds in Theorems 1.5, 1.8, and 1.9 and the 2-handle complexity of the 4-dimensional bordism in Theorem 3.9 are asymptotically optimal.

7.1 Lower Bounds of the Complexity of Lens Spaces

Recall that Theorem 1.14 in the introduction says the following: $c(L(n,1)) \in \Theta(n)$. *In fact, for each* $n > 3$,

$$\frac{n-3}{627\,419\,520} \leq c(L(n,1)) \leq n-3.$$

The upper bound in Theorem 1.14 is due to Jaco and Rubinstein [27]. In this section we give a proof of the lower bound.

For the proof, we need the value of the Cheeger-Gromov invariant of $L(n,1)$. For the finite fundamental group case, the Cheeger-Gromov invariants are determined by the Atiyah-Singer G-signatures [2] or the Atiyah-Patodi-Singer ρ-invariants [1]. In particular, for lens spaces, the computation in [1] can be reinterpreted as a computation of the Cheeger-Gromov invariant. We state a special case as a lemma, for use in this and the following subsections.

LEMMA 7.1. $\rho^{(2)}(L(n,1), \mathrm{id}_{\pi_1(L(n,1))}) = \dfrac{n}{3} + \dfrac{2}{3n} - 1.$

PROOF. Atiyah, Patodi, and Singer computed their ρ-invariant for general (including high dimensional) lens spaces [1, prop. 2.12]. For $L(n,1)$ and the regular representation α of $\pi_1(L(n,1)) = \mathbb{Z}_d$, their formula gives the following:

$$\rho_\alpha(L(n,1)) = \sum_{k=1}^{n-1} \cot^2\left(\frac{\pi k}{n}\right).$$

By the cotangent formula for the Dedekind sum (e.g., see [45]), the above sum is equal to $4n \sum_{k=1}^{n-1} ((k/n))^2$, where $((k/n))$ denotes the sawtooth function, whose value is $k/n - 1/2$ in our case. From this we obtain

$$\rho_\alpha(L(n,1)) = \frac{n^2}{3} + \frac{2}{3} - n.$$

Since $\frac{1}{n}\dim_{\mathbb{C}} = \dim^{(2)}_{\mathbb{Z}_n}$, we have $\rho^{(2)}(L(n,1), \mathrm{id}_{\pi_1(L(n,1))}) = \frac{1}{n}\rho_\alpha(L(n,1))$. From this Lemma 7.1 follows. □

PROOF OF THE LOWER BOUND IN THEOREM 1.14. We may assume $n > 0$ by reversing the orientation if $n < 0$. By Lemma 7.1 and Corollary 1.11, it follows that

$$c(L(n,1)) \geq \frac{1}{627\,419\,520}\left(n + \frac{2}{n} - 3\right) \geq \frac{n-3}{627\,419\,520}. \qquad \square$$

As discussed below in detail, it turns out that for odd n, the lower bound in Theorem 1.14 can be arbitrarily larger than lower bounds from previously known methods. Recall that for two functions $f(n)$ and $g(n)$, we say $g(n)$ is *dominated by* $f(n)$ and write $g(n) \in o(f(n))$ if $\limsup_{n\to\infty} |g(n)/f(n)| = 0$.

(1) Since $L(n,1)$ is a Seifert fibered space, the lower bound from the hyperbolic volume [42] does not apply to $L(n,1)$.
(2) When n is odd, since $H_1(L(n,1);\mathbb{Z}_2) = 0$, the methods of Jaco-Rubinstein-Tillmann [28–30] using double covers and the $\mathbb{Z}_2$-Thurston norm do not give any nonzero lower bound.
(3) In [41], Matveev and Pervova proved the following:

$$c(M) \geq 2\log_5 |tH_1(M)| + \operatorname{rank}_{\mathbb{Z}} H_1(M),$$

where $|tH_1(M)|$ denotes the order of the torsion subgroup of $H_1(M)$. For $M = L(n,1)$, this gives us $c(L(n,1)) \geq 2\log_5 n$. This bound is logarithmic, which is dominated by the linear lower bound in Theorem 1.14.
(4) In [41], they showed that $c(M) \geq c(\pi_1(M))$, where the complexity $c(G)$ of a group G is defined to be the minimal lengths of a finite presentation of G. The length of a finite presentation is the sum of the word length of the defining relators. Computation of $c(G)$ is difficult in general; even for $G = \mathbb{Z}_n$, the answer seems complicated. From the presentation $\langle g \mid g^n\rangle$, we obtain $c(\mathbb{Z}_n) \leq n$. Interestingly, for infinitely many n, $c(\mathbb{Z}_n)$ is much smaller than n. For instance, let $n = k^2 - 1$. Then $\mathbb{Z}_n$ admits a presentation $\langle x, y \mid x^k y^{-1}, x^{-1} y^k\rangle$. Since its length is $2(k+1)$, we have $c(\mathbb{Z}_n) \leq 2(k+1) = 2(\sqrt{n+1}+1)$. It follows that, for $M = L(n,1)$ with $n = k^2 - 1$, the lower bound $c(\pi_1(M))$ gives us at best $c(L(n,1)) \geq 2(\sqrt{n+1}+1)$. This is dominated by the linear lower bound in Theorem 1.14.

From the above observations, Theorem 1.12 in the introduction follows immediately.

Remark 7.2.

(1) In [7] we show that there are closed hyperbolic 3-manifolds (with fixed first homology) for which the complexity lower bounds obtained from Cheeger-Gromov invariants can be arbitrarily larger than the lower bound from the hyperbolic volume.

(2) There are closed 3-manifolds M such that the Cheeger-Gromov invariant $\rho^{(2)}(M,\phi)$, and consequently the lower bound of $c(M)$ given in Corollary 1.11, can be arbitrarily larger than the Thurston norm of any generator of $H^1(M;\mathbb{Z})$. For instance, the computational method in [18, prop. 3.2] tells us how to construct a satellite knot with a fixed genus, say g, whose zero surgery manifold M admits an arbitrarily large value of $\rho^{(2)}(M,\phi)$; the generator of $H^1(M)\cong\mathbb{Z}$ has Thurston norm $\leq 2g-1$.

7.2 Linear Cheeger-Gromov Bounds Are Optimal

By considering the case of lens spaces, we will prove Theorem 1.6, which says that the linear Cheeger-Gromov bound in Theorem 1.5 is asymptotically optimal. Recall from the introduction that we define $B^{\mathrm{sc}}(n)$ to be the optimal Cheeger-Gromov bound for 3-manifolds with simplicial complexity n, that is,

$$B^{\mathrm{sc}}(n) = \sup\left\{ |\rho^{(2)}(M,\phi)| \;\middle|\; \begin{array}{l} M \text{ has simplicial complexity} \leq n \\ \text{and } \phi \text{ is a homomorphism of } \pi_1(M) \end{array} \right\}.$$

Theorem 1.6 claims that

$$\limsup_{n\to\infty} \frac{B^{\mathrm{sc}}(n)}{n} \geq \frac{1}{288},$$

and consequently $B^{\mathrm{sc}}(n)\in\Omega(n)$.

PROOF OF THEOREM 1.6. Let s_n be the simplicial complexity of $L(n,1)$. By Lemma 7.1, $B^{\mathrm{sc}}(s_n)\geq \frac{1}{3}n-1$. Also, since $L(n,1)$ is obtained by surgery along the n-framed unknot, we have $s_n\leq 96n$ by Theorem 6.3. It follows that

$$\frac{B^{\mathrm{sc}}(s_n)}{s_n} \geq \frac{1}{288} - \frac{1}{s_n}. \tag{7.1}$$

Also, $s_n\geq c(L(n,1))\geq (n-3)/627\,419\,520$ by Theorem 1.14. So $s_n\to\infty$ as $n\to\infty$. It follows that (7.1) holds for infinitely many values of s_n. Taking lim sup of (7.1), the claimed inequality is obtained. □

We can also show that the Cheeger-Gromov bounds in Theorem 1.8 and 1.9 are asymptotically optimal. To state it formally, we use the following definitions.

DEFINITION 7.3. Define

$$B^{\mathrm{HL}}(\ell) = \sup\left\{ |\rho^{(2)}(M,\phi)| \;\middle|\; \begin{array}{l} M \text{ has Heegaard-Lickorish complexity} \leq \ell \\ \text{and } \phi \text{ is a homomorphism of } \pi_1(M) \end{array} \right\}.$$

For a framed link L, let $n(L)$ be the number of split unknotted zero framed components of L. As in [6], define the *surgery complexity* of a closed 3-manifold M

to be the minimum of $2c(L) + f(L) + n(L)$ over all framed links L in $\mathbb{S}^3$ from which M is obtained by surgery. Define

$$B^{\mathrm{surg}}(k) = \sup \left\{ |\rho^{(2)}(M,\phi)| \,\middle|\, \begin{array}{l} M \text{ has surgery complexity} \le k \\ \text{and } \phi \text{ is a homomorphism of } \pi_1(M) \end{array} \right\}.$$

Theorems 1.8 and 1.9 tell us that $B^{\mathrm{HL}}(\ell) \in O(\ell)$ and $B^{\mathrm{surg}}(k) \in O(k)$.

THEOREM 7.4. *$B^{\mathrm{HL}}(\ell) \in \Omega(\ell)$ and $B^{\mathrm{surg}}(k) \in \Omega(k)$. In fact,*

$$\frac{1}{3} \le \limsup_{\ell\to\infty} \frac{B^{\mathrm{HL}}(\ell)}{\ell} \le 251\,258\,280$$

and

$$\frac{1}{3} \le \limsup_{k\to\infty} \frac{B^{\mathrm{surg}}(k)}{k} \le 34\,856\,640.$$

PROOF. The upper bounds are immediately obtained from Theorems 1.8 and 1.9. The proofs of the lower bounds are identical with that of Theorem 1.6; instead of the fact that the simplicial complexity of $L(n,1)$ is not greater than $96n$, we use that both the Heegaard-Lickorish complexity and the surgery complexity of $L(n,1)$ are not greater than n. This gives us the lower bound $\frac{1}{3}$ of the lim sup instead of $\frac{1}{3\cdot 96} = \frac{1}{288}$. □

7.3 Bordisms with Linear 2-Handle Complexity Are Optimal

Finally, we show that the 2-handle complexity $195 \cdot d(\zeta_M) + 975 \cdot d(u)$ in Theorem 3.9 is asymptotically the best possible. For the reader's convenience, we recall Theorem 3.9: *suppose M is a closed 3-manifold endowed with a triangulation of complexity $d(\zeta_M)$. Suppose M is over G via a simplicial-cellular map $\phi : M \to BG$. If there is a 4-chain $u \in C_4(BG)$ satisfying $\partial u = \phi_\#(\zeta_M)$, then there exists a smooth bordism W between M and a trivial end such that 2-handle complexity of W is at most $195 \cdot d(\zeta_M) + 975 \cdot d(u)$.* Here $\zeta_M \in C_3(M)$ is the sum of 3-simplices that represents the fundamental class of M.

To state our result, we formally define "the best possible 2-handle complexity" as a function in $k := d(\zeta_M) + d(u)$ as follows:

DEFINITION 7.5. Let $\mathcal{M}(k)$ be the collection of pairs (M,ϕ) of a closed triangulated 3-manifold M and a simplicial-cellular map $\phi : M \to BG$ admitting a 4-chain $u \in C_4(BG)$ such that $\partial u = \phi_\#(\zeta_M)$ and $k = d(\zeta_M) + d(u)$. For a given (M,ϕ), let $\mathcal{B}(M,\phi)$ be the collection of bordisms W over G between M and a trivial end. Define

$$B^{2\mathrm{h}}(k) := \sup_{(M,\phi)\in\mathcal{M}(k)} \ \min_{W\in\mathcal{B}(M,\phi)} \{\text{2-handle complexity of } W\}.$$

Briefly speaking, $B^{2\mathrm{h}}(k)$ is the optimal (smallest) value for which the following holds: for any (M,ϕ) in $\mathcal{M}(k)$ there is a desired bordism W with 2-handle complexity not greater that $B^{2\mathrm{h}}(k)$.

THEOREM 7.6. *$B^{2h}(k) \in O(k) \cap \Omega(k)$. In fact,*

$$\frac{1}{107\,712} \leq \limsup_{k\to\infty} \frac{B^{2h}(k)}{k} \leq 975.$$

PROOF. Theorem 3.9 tells us that 975 is an upper bound of $B^{2h}(k)/k$. Consequently, $B^{2h}(k) \in O(k)$.

To show the remaining conclusion, we consider the lens space $M = L(n, 1)$ and $G = \mathbb{A}^3(\mathbb{Z}_n)$. By Theorem 6.3, there is a triangulation of M of simplicial complexity at most $96n$. That is, $d(\zeta_M) \leq 96n$. Appealing to Theorem 3.7, choose a simplicial-cellular map $\phi : M \to B\mathbb{A}^3(\mathbb{Z}_n)$ that induces the inclusion $\pi_1(M) = \mathbb{Z}_n \to \mathbb{A}^3(\mathbb{Z}_n)$ defined in Definition 5.1. Similarly to the proof of Theorem 6.1, there is a 4-chain $u \in C_4(BG)$ such that $\partial u = \phi_\#(\zeta_M)$ and $d(u) \leq 186d(\zeta_M)$, by Theorem 5.2. Let $k = d(\zeta_M) + d(u)$. By definition, $(M, \phi) \in \mathcal{M}(k)$. Also note that

$$k \leq 187d(\zeta_M) \leq 17\,952n.$$

We claim that

$$\min_{W\in\mathcal{B}(M,\phi)} \{\text{2-handle complexity of } W\} \geq \frac{k}{107\,712} - \frac{1}{2}.$$

To show the claim, suppose W is a bordism over G between $M = L(n, 1)$ and a trivial end. Then we can compute $\rho^{(2)}(M, \phi)$ as the L^2-signature defect of W. In particular, if W has 2-handle complexity r, then $|\rho^{(2)}(M, \phi)| \leq 2r$. By the L^2-induction property and by Lemma 7.1, we have

$$\rho^{(2)}(M, \phi) = \rho^{(2)}(M, \mathrm{id}_{\pi_1(M)}) = \frac{n}{3} + \frac{2}{3n} - 1.$$

Combining these, we obtain

$$r \geq \frac{n}{6} - \frac{1}{2} \geq \frac{k}{107\,712} - \frac{1}{2}$$

as claimed.

From the claim, it follows that

$$B^{2h}(k) \geq \frac{k}{107\,712} - \frac{1}{2}. \tag{7.2}$$

Obviously $k \geq d(\zeta_M) \geq c(L(n, 1))$, and by Theorem 1.14, $c(L(n, 1)) \to \infty$ as $n \to \infty$. It follows that (7.2) holds for infinitely many k. This completes the proof. $\square$

Appendix A Simplicial Sets and Simplicial Classifying Spaces

In this appendix we give a quick review of basic definitions and facts on simplicial sets for readers not familiar with them, focusing on those we needed in this paper, and present a detailed proof of Theorem 3.7 stated in the body. (See Proposition A.1.) There are numerous excellent references on simplicial sets. For instance, [25, 43] provide thorough extensive treatements, and [24] is an easily accesible introduction for nonexperts.

A.1 Simplicial Sets and Geometric Realizations

We begin with a formal definition of a simplicial set. A *simplicial set* X is a collection $\{X_0, X_1, \dots\}$ of sets X_n together with functions $d_i : X_n \to X_{n-1}$ ($n = 1, 2, \dots, i = 0, \dots, n$) and $s_i : X_n \to X_{n+1}$ ($n = 0, 1, \dots, i = 0, \dots, n$) satisfying the following:

$$\begin{aligned} &d_i d_j = d_{j-1} d_i \quad \text{if } i < j, && d_i s_j = s_j d_{i-1} \quad \text{if } i > j+1, \\ &d_i s_j = s_{j-1} d_i \quad \text{if } i < j, && s_i s_j = s_{j+1} s_i \quad \text{if } i \le j, \\ &d_j s_j = d_{j+1} s_j = \text{id}. \end{aligned} \tag{A.1}$$

An element $\sigma \in X_n$ is called an *n-simplex* of X, and d_i and s_i are called the *face map* and *degeneracy map*. A simplex $\sigma \in X_n$ is called *degenerate* if $\sigma = s_i \tau$ for some i and $\tau \in X_{n-1}$.

A *morphism* $f : X \to Y$ of simplicial sets is defined to be a collection of maps $f : X_n \to Y_n$ satisfying $f d_i = d_i f$ and $f s_i = s_i f$. Simplicial sets and their morphisms form a category, which we denote by **sSet**.

The underlying geometric picture is as follows. Define the standard n-simplex Δ^n to be the convex hull $[e_1, \dots, e_n]$ of the standard basis in $\mathbb{R}^{n+1}$. Then the face map d_i is an incarnation of taking the i^{th} face $[e_1, \dots, \widehat{e_i}, \dots, e_n]$ of Δ^n by omitting the i^{th} vertex; similarly s_i corresponds to producing a degenerate $(n+1)$-simplex $[e_1, \dots, e_i, e_i, \dots, e_n]$ from Δ^n by repeating the i^{th} vertex. It is straightforward to verify the above relations of the d_i and s_i for the case of Δ^n. As the key information of a simplicial set, the maps d_i and s_i indicate how the simplices are assembled in the geometric picture: for an n-simplex σ and an $(n-1)$-simplex τ, $d_i \sigma = \tau$ corresponds to an identification of τ with the i^{th} face of σ, and similarly, $s_i \tau = \sigma$ corresponds to an identification of σ with τ via a collapsing.

The above geometric idea is formalized to the following definition of the *geometric realization* $|X|$ of a simplicial set X. Let $D_i : \Delta^n \to \Delta^{n+1}$ be the i^{th} face inclusion, i.e., the affine map determined by $(e_0, \dots, e_n) \to (e_0, \dots, \widehat{e_i}, \dots, e_{n+1})$. Let $S_i : \Delta^{n+1} \to \Delta^n$ be the projection onto the i^{th} face; i.e., the affine map determined by $(e_0, \dots, e_{n+1}) \to (e_0, \dots, e_i, e_i, \dots, e_n)$. Then

$$|X| := \left(\coprod_{n \ge 0} X_n \times \Delta^n \right) \Big/ \sim$$

where the equivalence relation $\sim$ is generated by $(\sigma, D_i(p)) \sim (d_i(\sigma), p)$ for $\sigma \in X_{n+1}$ and $p \in \Delta^n$, $(\sigma, S_i(p)) \sim (s_i(\sigma), p)$ for $\sigma \in X_n$ and $p \in \Delta^{n+1}$.

Due to Milnor [44], the space $|X|$ is a CW-complex whose n-cells are in 1-1 correspondence to nondegenerate n-simplices of X; if $\sigma \in X_n$ is nondegenerate, the characteristic map of the corresponding n-cell (which we call an n-simplex of

$|X|$) is given by

$$\varphi_\sigma : \Delta^n = \{\sigma\} \times \Delta^n \hookrightarrow \coprod_{n \geq 0} X_n \times \Delta^n \xrightarrow{q} |X|.$$

From this it follows that $|X|$ is a simplicial-cell complex in the sense of Definition 3.6 in the body of the paper.

A morphism $f : X \to Y$ of simplicial sets gives rise to a continuous map $|f| : |X| \to |Y|$ induced by $\{\sigma\} \times \Delta^n \xrightarrow{\mathrm{id}} \{f(\sigma)\} \times \Delta^n$:

$$\begin{tikzcd}
\{\sigma\} \times \Delta \arrow[r, hook] \arrow[d, "\mathrm{id}"'] & \coprod_{n \geq 0} X_n \times \Delta^n \arrow[r, "q"] \arrow[d, "f \times \mathrm{id}"'] & {|X|} \arrow[d, "{|f|}"'] \\
\{f(\sigma)\} \times \Delta \arrow[r, hook] & \coprod_{n \geq 0} Y_n \times \Delta^n \arrow[r, "q"'] & {|Y|}
\end{tikzcd}.$$

We remark that even when $\sigma \in X_n$ is nondegenerate, $f(\sigma) \in Y_n$ may be degenerate: $q(\{f(\sigma)\} \times \Delta^n)$ may be a k-simplex in $|Y|$ with $k < n$.

From the above diagram, it follows that $|f|$ is a simplicial-cellular map in the sense of Definition 3.6 in the body of the paper.

A.2 Chain Complexes

A based chain complex $\mathbb{Z}X_*$ called the (unnormalized) *Moore complex* is naturally associated to a simplicial set X, similarly to the construction for an ordered simplicial complex: define $\mathbb{Z}X_n$ to be the free abelian group generated by X_n, and define the boundary map $\partial : \mathbb{Z}X_n \to \mathbb{Z}X_{n-1}$ by $\partial(\sigma) = \sum_{i=0}^n (-1)^n d_i(\sigma)$ for an n-simplex $\sigma \in X_n$. Then $(\mathbb{Z}X_*, \partial)$ becomes a based chain complex with the n-simplices as basis elements. This gives rise to a functor **sSet** $\to$ $\mathbf{Ch}_+^b$ to the category $\mathbf{Ch}_+^b$ of positive based chain complexes.

We remark that the chain complex $\mathbb{Z}X_*$ of a simplicial set is distinct from the cellular chain complex $C_*(X) := C_*(|X|)$ of its realization $|X|$, since degenerate simplices are still generators of $\mathbb{Z}X_*$, while they do not give a cell of $|X|$.

The chain complexes $\mathbb{Z}X_*$ and $C_*(X)$ are related as follows. Let $D_*(X)$ be the subgroup of $\mathbb{Z}X_*$ generated by degenerate simplices of X, that is, simplices of the form $s_i\tau$ for some other simplex τ. It is known that $D_*(X)$ is a contractible subcomplex and $C_*(X) \cong \mathbb{Z}X_*/D_*(X)$. Consequently, we have a short exact sequence

$$0 \to D_*(X) \to \mathbb{Z}X_* \xrightarrow{p} C_*(X) \to 0$$

where the projection p is a chain homotopy equivalence. We remark that the essential reason is that the n-cells of the CW complex $|X|$ are in 1-1 correspondence with the nondegenerate n-simplices of the simplicial set X. For a proof, see [43, sec. 22] or [38, p. 236].

We note that the projection $p : \mathbb{Z}X_* \to C_*(X)$ is a natural transformation between the functors $\mathbb{Z}(-)_*$, $C_*(-) : \mathbf{sSet} \to \mathbf{Ch}^b_+$. That is, if $\phi : X \to Y$ is a morphism of simplicial sets, then $p\phi = \phi_\# p$.

We also note that if X is an (ordered) simplicial complex that is viewed as a simplicial set, then $C_*(X)$ can be viewed as a subcomplex of $\mathbb{Z}X_*$; for, in this case, the i^{th} face $d_i\sigma$ of a nondegenerate simplex σ is nondegenerate, and consequently the nondegenerate simplices generate a subcomplex of $\mathbb{Z}X_*$ that can be identified with $C_*(X)$. We remark that it does not hold for an arbitrary simplicial set X; as an exercise, such an example can be easily obtained using the simplicial classifying space BG discussed in A.4.

A.3 Products

One of the technical advantages of simplicial sets (in particular allowing degenerate simplices) is that the product construction is simple. For two simplicial sets X and Y, $X \times Y$ is defined by $(X \times Y)_n := X_n \times Y_n$; together with $d_i(\sigma,\tau) = (d_i\sigma, d_i\tau)$ and $s_i(\sigma,\tau) = (s_i\sigma, s_i\tau)$, $X \times Y$ becomes a simplicial set.

A.4 Simplicial Classifying Spaces

Let G be a group. The *simplicial classifying space BG* is defined by the bar construction: BG is the simplicial set with $BG_n = \{[g_1,\dots,g_n] \mid g_i \in G\}$ (in particular, $BG_0 = \{[\,]\}$ consists of one element) where the face map $d_i : BG_n \to BG_{n-1}$ and the degeneracy map $s_i : BG_n \to BG_{n+1}$ are given by

$$d_i[g_1,\dots,g_n] = \begin{cases} [g_2,\dots,g_n] & \text{if } i = 0, \\ [g_1,\dots,g_{i-1}, g_ig_{i+1}, g_{i+2},\dots,g_n] & \text{if } 0 < i < n, \\ [g_1,\dots,g_{n-1}] & \text{if } i = n, \end{cases}$$
$$s_i[g_1,\dots,g_n] = [g_1,\dots,g_i,e,g_{i+1},\dots,g_n].$$

From the definition, it is straightforward to verify that $B : \mathbf{Gp} \to \mathbf{sSet}$ is a functor of the category of groups $\mathbf{Gp}$. It is well known that the geometric realization $|BG|$ of BG is an Eilenberg-MacLane space $K(G,1)$.

In the following statement, $\pi_1(A)$ of a space A is understood as the free product of the fundamental groups of the path components.

PROPOSITION A.1. *Suppose X is a simplicial set and $\phi : \pi_1(|X|) \to G$ is a group homomorphism. Then there is a morphism $f : X \to BG$ of simplicial sets such that $|f|_* : \pi_1(|X|) \to \pi_1(|BG|) = G$ is equal to ϕ.*

We remark that Theorem 3.7 in the body of the paper is an immediate consequence of Proposition A.1.

PROOF OF PROPOSITION A.1. We will define f on X_n inductively and check the functoriality $d_if = fd_i$ and $s_if = fs_i$ at each step.

We start by defining f on X_0 by $f(v) = [\,] \in BG_0$ for any $v \in X_0$. For each 0-simplex v of X, choose a path γ_v to it from the basepoint of its component in $|X|$.

(For example, one may take a spanning forest of the 1-skeleton to determine the γ_v.) For $\sigma \in X_1$ from $w := d_1\sigma$ to $v := d_0\sigma$, we define

$$f(\sigma) = \big[\phi\big(\gamma_w \cdot \psi_\sigma \cdot \gamma_v^{-1}\big)\big] \in BG_1.$$

We have that $f(d_i\sigma) = [\,] = d_i f(\sigma)$ for $\sigma \in X_1$ and $f(s_i\tau) = s_i[\,] = s_i f(\tau)$ for $\tau \in X_0$. Also note that $f(\sigma) = [e]$ when σ is a degenerate 1-simplex (that is, $\sigma = s_i\sigma'$ for some $\sigma' \in X_0$).

For notational convenience, for $\sigma = [g_1, \dots, g_k] \in BG_k$ we often denote by σ the sequence $g_1, \dots, g_k$ obtained by removing the brackets. In particular, if $\sigma \in BG_k$ and $\tau \in BG_\ell$, then $[\sigma, \tau]$ denotes an element in $BG_{k+\ell}$.

For $\sigma \in X_2$, define

$$f(\sigma) = [f(d_2\sigma), f(d_0\sigma)] \in BG_2.$$

Note that we have $f(d_0\sigma) \cdot f(d_1\sigma)^{-1} \cdot f(d_2\sigma) = e$ in G since $\partial\sigma = d_0\sigma - d_1\sigma + d_2\sigma$. Using this we check the functoriality: for $\sigma \in X_2$ and $\tau \in X_1$,

$$\begin{aligned}
d_0 f(\sigma) &= d_0[f(d_2\sigma), f(d_0\sigma)] = [f(d_0\sigma)] = f(d_0\sigma),\\
d_1 f(\sigma) &= d_1[f(d_2\sigma), f(d_0\sigma)] = [f(d_2\sigma) f(d_0\sigma)] = [f(d_1\sigma)] = f(d_1\sigma),\\
d_2 f(\sigma) &= d_2[f(d_2\sigma), f(d_0\sigma)] = [f(d_2\sigma)] = f(d_2\sigma),\\
f(s_0\tau) &= [f(d_2s_0\tau), f(d_0s_0\tau)] = [f(s_0d_1\tau), f(\tau)] = [e, f(\tau)] = s_0 f(\tau),\\
f(s_1\tau) &= [f(d_2s_1\tau), f(d_0s_1\tau)] = [f(\tau), f(s_0d_0\tau)] = [f(\tau), e] = s_1 f(\tau).
\end{aligned}$$

In general, suppose f has been defined on X_k for $k < n$. For $\sigma \in X_n$ we define f by

$$f(\sigma) = \big[f(d_n\sigma), f\big(d_0^{n-1}\sigma\big)\big]. \tag{A.2$_n$}$$

We claim that

$$f(\sigma) = [f(d_2 \cdots d_n\sigma), f(d_0\sigma)]. \tag{A.3$_n$}$$

For it obviously holds when $n = 2$; for $n > 2$, using (A.2)$_{n-1}$ and (A.3)$_{n-1}$ as induction hypotheses, we obtain

$$\begin{aligned}
f(\sigma) &= [f(d_n\sigma), f(d_0^{n-1}\sigma)] && \text{by (A.2)}_n\\
&= [f(d_2 \cdots d_{n-1}(d_n\sigma)), f(d_0d_n\sigma), f(d_0^{n-1}\sigma)] && \text{by (A.3)}_{n-1}\\
&= [f(d_2 \cdots d_{n-1}d_n\sigma), f(d_{n-1}d_0\sigma), f(d_0^{n-1}\sigma)] && \text{by (A.1)}\\
&= [f(d_2 \cdots d_{n-1}d_n\sigma), f(d_0\sigma)] && \text{by (A.2)}_{n-1}.
\end{aligned}$$

Now using (A.1), (A.2), and (A.3) we verify the functoriality: for $\sigma \in X_n$, if $i < n-1$, we have

$$\begin{aligned}
d_i f(\sigma) &= d_i\big[f(d_n\sigma), f\big(d_0^{n-1}\sigma\big)\big] = \big[d_i f(d_n\sigma), f\big(d_0^{n-1}\sigma\big)\big]\\
&= \big[f(d_id_n\sigma), f\big(d_0^{n-1}\sigma\big)\big] = \big[f(d_{n-1}d_i\sigma), f\big(d_0^{n-2}d_i\sigma\big)\big] = f(d_i\sigma),
\end{aligned}$$

and if $i > 1$, we have

$$d_i f(\sigma) = d_i[f(d_2 \cdots d_n\sigma), f(d_0\sigma)] = [f(d_2 \cdots d_n\sigma), d_{i-1} f(d_0\sigma)] = \\ [f(d_2 \cdots d_n\sigma), f(d_{i-1}d_0\sigma)] = [f(d_2 \cdots d_{n-1}d_i\sigma), f(d_0 d_i\sigma)] = f(d_i\sigma).$$

So, in any case, we have $d_i f(\sigma) = f(d_i\sigma)$. Also, for $\tau \in X_{n-1}$, if $i < n-1$, we have

$$s_i f(\tau) = s_i\big[f(d_{n-1}\tau), f\big(d_0^{n-2}\tau\big)\big] = \big[s_i f(d_{n-1}\tau), f\big(d_0^{n-2}\tau\big)\big] \\ = \big[f(s_i d_{n-1}\tau), f\big(d_0^{n-2}\tau\big)\big] = \big[f(d_n s_i\tau), f\big(d_0^{n-1}s_i\tau\big)\big] = f(s_i\tau),$$

and if $i > 0$, we have

$$s_i f(\tau) = s_i[f(d_2 \cdots d_{n-1}\tau), f(d_0\tau)] = [f(d_2 \cdots d_{n-1}\tau), s_{i-1} f(d_0\tau)] = \\ [f(d_2 \cdots d_{n-1}\tau), f(s_{i-1}d_0\tau)] = [f(d_2 \cdots d_n s_i\tau), f(d_0 s_i\tau)] = f(s_i\tau).$$

This completes the proof that $f : X \to BG$ is a well-defined morphism of simplicial sets.

From the definition of f on X_1, it follows that f induces the given homomorphism $\phi : \pi_1(|X|) \to G$. □

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