

COMBINATORIAL TORSIONS AND THE COBORDISM GROUP OF HOMOLOGY CYLINDERS

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- Homology cylinder
- The homology cobordism group of homology cylinders
- Main Theorem
- The Torelli group analogue
- Outline of the proof

HOMOLOGY CYLINDER

- For $g, k \geq 0$, let $\Sigma_{g,k}$:= oriented compact surface of genus g with k boundary components.
- A **homology cylinder** (M, i_+, i_-) over $\Sigma_{g,k}$ is a 3-manifold M together with two embeddings $i_+, i_- : \Sigma_{g,k} \rightarrow \partial M$ such that
 - (1) i_+ is orientation preserving and i_- is orientation reversing,
 - (2) $\partial M = i_+(\Sigma_{g,k}) \cup i_-(\Sigma_{g,k})$ and
$$i_+(\Sigma_{g,k}) \cap i_-(\Sigma_{g,k}) = i_+(\partial\Sigma_{g,k}) = i_-(\partial\Sigma_{g,k}),$$
 - (3) $i_+|_{\partial\Sigma_{g,k}} = i_-|_{\partial\Sigma_{g,k}}$,
 - (4) $i_+, i_- : H_*(\Sigma_{g,k}; \mathbb{Z}) \rightarrow H_*(M; \mathbb{Z})$ are isomorphisms.

HOMOLOGY CYLINDER: EXAMPLES

- $\mathcal{M}_{g,k}$:= the mapping class group.

For a diffeomorphism $\varphi \in \mathcal{M}_{g,k}$,

$$M(\varphi) = (\Sigma_{g,k} \times [0, 1] / \sim, i_+ = \text{id} \times 0, i_- = \varphi \times 1).$$

where \sim is given by $(x, s) \sim (x, t)$ for $x \in \partial\Sigma_{g,k}$ and $s, t \in [0, 1]$ is a homology cylinder.

- Let K be a knot of genus g such that $\Delta_K(t)$ is monic and such that $\deg(\Delta_K(t)) = 2g$. Let $\Sigma \subset S^3 - N(K)$ be a minimal genus Seifert surface. Then $(S^3 - N(K)) - \Sigma \times (0, 1)$ is a homology cylinder over $\Sigma_{g,1}$ in a natural way.

- Two homology cylinders (M, i_+, i_-) and (N, j_+, j_-) over $\Sigma_{g,k}$ are called **isomorphic** if there exists an orientation preserving diffeomorphism $f : M \rightarrow N$ satisfying $j_+ = f \circ i_+$ and $j_- = f \circ i_-$.
- $\mathcal{C}_{g,k} :=$ the **monoid** of all isomorphism classes of homology cylinders over $\Sigma_{g,k}$.
- The product operation on $\mathcal{C}_{g,k}$:

$$(M, i_+, i_-) \cdot (N, j_+, j_-) := (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-).$$

- Two homology cylinders (M, i_+, i_-) and (N, j_+, j_-) over $\Sigma_{g,k}$ are called **homology cobordant** if there exists a compact oriented smooth 4-manifold W such that

$$\partial W = M \cup (-N) / (i_+(x) = j_+(x), i_-(x) = j_-(x), x \in \Sigma_{g,k}),$$

and such that the inclusion induced maps $H_*(M; \mathbb{Z}) \rightarrow H_*(W; \mathbb{Z})$ and $H_*(N; \mathbb{Z}) \rightarrow H_*(W; \mathbb{Z})$ are isomorphisms.

- $\mathcal{H}_{g,k} :=$ the **group** of homology cobordism classes of elements in $\mathcal{C}_{g,k}$

- For a diffeomorphism $\varphi \in \mathcal{M}_{g,k}$,

$$M(\varphi) = (\Sigma_{g,k} \times [0, 1] / \sim, i_+ = \text{id} \times 0, i_- = \varphi \times 1).$$

where \sim is given by $(x, s) \sim (x, t)$ for $x \in \partial\Sigma_{g,k}$ and $s, t \in [0, 1]$ is a homology cylinder.

THEOREM (GAROUFALIDIS-LEVINE)

$\mathcal{M}_{g,k}$ embeds into $\mathcal{C}_{g,k}$ and $\mathcal{H}_{g,k}$.

- If $g \geq 3$, then $\mathcal{M}_{g,k}$ is perfect: $\mathcal{M}_{g,k} = [\mathcal{M}_{g,k}, \mathcal{M}_{g,k}]$.

THEOREM (2009, GODA-SAKASAI)

For $g \geq 1$, $\mathcal{C}_{g,1}$ surjects to \mathbb{Z}^∞ , hence not perfect.

- Idea of proof: rank of sutured Floer homology $SFH(M, i_+(\partial\Sigma_{g,1}))$.

Questions (Garoufalidis-Levine, Goda-Sakasai)

Is $\mathcal{H}_{g,k}$ infinitely generated? Is $\mathcal{H}_{g,k}$ not perfect?

- $\mathcal{H}_{0,0}^{\text{smooth}} \cong \mathcal{H}_{0,1}^{\text{smooth}} \cong \Theta_3^{\text{smooth}}$ and it has infinite rank.
- $\mathcal{H}_{0,0}^{\text{top}} \cong \mathcal{H}_{0,1}^{\text{top}} \cong \Theta_3^{\text{top}} = 0$
- $\mathcal{H}_{0,2} \cong \mathbb{Z} \oplus \text{Conc} \rightarrow (\mathbb{Z}/2)^\infty \oplus (\mathbb{Z}/4)^\infty \oplus \mathbb{Z}^\infty$

THEOREM (CHA-FRIEDL-K.)

(1) If $b_1(\Sigma_{g,k}) > 0$, then there exists an epimorphism

$$\mathcal{H}_{g,k} \rightarrow (\mathbb{Z}/2)^\infty$$

which splits. In particular, the abelianization of $\mathcal{H}_{g,k}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty$.

(2) If $k > 1$, then there exists an epimorphism

$$\mathcal{H}_{g,k} \rightarrow \mathbb{Z}^\infty.$$

Furthermore, the abelianization of $\mathcal{H}_{g,k}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty \oplus \mathbb{Z}^\infty$.

COROLLARY

*If $b_1(\Sigma_{g,k}) > 0$, then $\mathcal{H}_{g,k}$ is infinitely generated and not perfect.
Furthermore, $\mathcal{H}_{g,k}$ is not finitely related.*

THE TORELLI GROUP ANALOGUE

- The Torelli group $\mathcal{I}_{g,k}$ of $\mathcal{M}_{g,k}$ is

$$\mathcal{I}_{g,k} := \{g \in \mathcal{M}_{g,k} \mid g \text{ acts trivially on } H_1(\Sigma_{g,k})\}.$$

- For a homology cylinder (M, i_+, i_-) over Σ , we can define the automorphism

$$\varphi(M) := (i_+)_*^{-1} (i_-)_* : H_1(\Sigma) \xrightarrow[(i_-)_*]{\cong} H_1(M) \xrightarrow[(i_+)_*^{-1}]{\cong} H_1(\Sigma).$$

Then $\varphi(M) \in \text{Aut}^*(H) \subset \text{Aut}(H)$ for $H = H_1(\Sigma)$.

- The Torelli group of $\mathcal{H}_{g,k}$ is

$$\mathcal{IH}_{g,k} := \{(M, i_+, i_-) \in \mathcal{H}_{g,k} \mid \varphi(M) = \text{id}\}.$$

THEOREM

- (1) *The group $\mathcal{I}_{g,n}$ is torsion-free,*
- (2) *the group $\mathcal{I}_{g,n}$ is finitely generated for $g \geq 3$ and $n = 0, 1$,*
- (3) *the group $\mathcal{I}_{2,0}$ is a free group on infinitely many generators,*
- (4) *if $g \geq 3$, then the abelianization of $\mathcal{I}_{g,1}$ is isomorphic to $\mathbb{Z}^a \oplus (\mathbb{Z}/2)^b$ for some $a, b \in \mathbb{N}$.*

It is an open question whether the Torelli group $\mathcal{I}_{g,1}$ is finitely related for $g \geq 3$.

THEOREM (MORITA)

The abelianization of $\mathcal{IH}_{g,1}$ has infinite rank.

THEOREM (CHA-FRIEDL-K.)

(1) If $b_1(\Sigma_{g,k}) > 0$, then there exists an epimorphism

$$\mathcal{IH}_{g,k} \rightarrow (\mathbb{Z}/2)^\infty$$

which splits. In particular, the abelianization of $\mathcal{IH}_{g,k}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty$.

(2) If $g > 1$ or $k > 1$, then there exists an epimorphism

$$\mathcal{IH}_{g,k} \rightarrow \mathbb{Z}^\infty.$$

Furthermore, the abelianization of $\mathcal{IH}_{g,k}$ contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty \oplus \mathbb{Z}^\infty$.

COROLLARY

If $b_1(\Sigma_{g,k}) > 0$, then $\mathcal{IH}_{g,k}$ is infinitely generated and not perfect. Furthermore, $\mathcal{IH}_{g,k}$ is not finitely related.

STRATEGY FOR PROVING MAIN THEOREM

- (1) Let $H := H_1(\Sigma_{g,k})$. Construct a homomorphism

$$\mathcal{H}_{g,k} \rightarrow Q(H)^\times / \sim$$

for some quotient group of $Q(H)^\times$, the field of fractions of $\mathbb{Z}[H]$, using the [Reidemeister torsion](#).

- (2) Show that $\mathcal{H}_{g,k}$ has (the desired) nontrivial image in the abelian group $Q(H)^\times / \sim$ under the homomorphism.

- $H := H_1(\Sigma_{g,k})$ and $Q(H) :=$ the quotient field of $\mathbb{Z}[H]$.
- For a homology cylinder (M, i_+, i_-) , let $\Sigma_{\pm} = i_{\pm}(\Sigma) \subset M$.
- Considering $\pi_1(M) \rightarrow H_1(M) \xleftarrow{\cong} H_1(\Sigma_+) \xleftarrow{i_+} H$, the torsion of (M, i_+, i_-) is defined by

$$\tau(M) = \tau(M, \Sigma_+; Q(H)) := \tau(C_*(\tilde{M}, i_+(\widetilde{\Sigma_{g,k}}); \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(M)]} Q(H)).$$

- For any homology cylinder M , we have $\tau(M) = \text{ord } H_1(M, \Sigma_+; \mathbb{Z}[H])$.
- A function $\mathcal{C}_{g,k} \xrightarrow{\tau} Q(H)^{\times} / \pm H$.

CONSTRUCTION OF $\mathcal{H}_{g,k} \rightarrow Q(H)^\times / \sim$

THEOREM

Let $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$ be homology cylinders over Σ .

Then

$$\tau(M \cdot N) \doteq \tau(M) \cdot \varphi(M)(\tau(N)).$$

THEOREM

Let $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$ be homology cylinders over a surface Σ which are *homology cobordant*. Write $H = H_1(\Sigma)$. Then

$$\tau(M) \doteq \tau(N) \cdot q \cdot \bar{q} \in Q(H)^\times$$

for some $q \in Q(H)^\times$.

CONSTRUCTION OF $\mathcal{H}_{g,k} \rightarrow Q(H)^\times / \sim$

- $A(H) := \{\pm h \cdot p^{-1} \cdot \varphi(p) \mid h \in H, p \in Q(H)^\times, \text{ and } \varphi \in \text{Aut}^*(H)\}$.
- $N(H) := \{\pm h \cdot q \cdot \bar{q} \mid q \in Q(H)^\times, h \in H\}$.
- Denote $A(H)N(H)$ by AN .

THEOREM

The torsion invariant gives rise to a group homomorphism

$$\tau: \mathcal{H}_{g,k} \rightarrow Q(H)^\times / AN,$$

where $H = H_1(\Sigma_{g,k})$.

- In fact, there is a homomorphism

$$\mathcal{H}_{g,k} \rightarrow \text{Aut}^*(H) \rtimes Q(H)^\times / N(H).$$

STRUCTURE OF $Q(H)^\times / AN$

- $Q(H)^{sym} = \{p \in Q(H)^\times \mid p = \bar{p} \text{ in } Q(H)^\times / A\}$.
- There is an exact sequence:

$$1 \rightarrow \frac{Q(H)^{sym}}{AN} \rightarrow \frac{Q(H)^\times}{AN} \rightarrow \frac{Q(H)^\times}{Q(H)^{sym}} \rightarrow 1$$

- For $p, q \in \mathbb{Z}[H]$, define $p \sim q$ if $p = \varphi(q)$ for some $\varphi \in \text{Aut}^*(H)$.
- p is called **self-dual** if $p \sim \bar{p}$.
- For $p \in Q(H)$ and an irreducible $\lambda \in \mathbb{Z}[H]$,
 $e_\lambda(p)$ = the sum of exponents of distinct irreducible factors μ of p such that $\mu \sim \lambda$.

STRUCTURE OF $Q(H)^\times / AN$

- For each **self-dual** irreducible $\lambda \in \mathbb{Z}[H]$, define

$$\Psi_\lambda: Q(H)^{sym} / AN \rightarrow \mathbb{Z}/2$$

by $\Psi_\lambda(p \cdot AN) = e_\lambda(p) \pmod{2}$.

- For each **non-self-dual** irreducible $\mu \in \mathbb{Z}[H]$, define

$$\Theta_\mu: Q(H)^\times / Q(H)^{sym} \rightarrow \mathbb{Z}$$

by $\Theta_\mu(p \cdot Q(H)^{sym}) = e_\mu(p) - e_{\bar{\mu}}(p)$.

STRUCTURE OF $Q(H)^\times / AN$

THEOREM

$$\begin{aligned} (1) \quad \bigoplus_{[\lambda]} \Psi_\lambda & : Q(H)^{sym} / AN \xrightarrow{\downarrow \mathbb{R}} \bigoplus_{[\lambda]} \mathbb{Z}/2, \\ (2) \quad \bigoplus_{\{[\mu], [\bar{\mu}]\}} \Theta_\mu & : Q(H)^\times / Q(H)^{sym} \xrightarrow{\downarrow \mathbb{R}} \bigoplus_{\{[\mu], [\bar{\mu}]\}} \mathbb{Z}. \end{aligned}$$

LEMMA (SYMMETRY)

Suppose Σ is a surface with at most one boundary components. Then for any homology cylinder M over Σ , we have $\tau(M) = \overline{\tau(M)}$ in $Q(H)^\times / AN$.

Remark

If Σ has more than one boundary component, $\tau(M)$ is in general **asymmetric even modulo AN** . Using this, we can prove that the image of $\tau: \mathcal{H}_{g,k} \rightarrow Q(H)^\times / AN$ contains a subgroup isomorphic to \mathbb{Z}^∞ .