Splittings of von Neumann rho–invariants

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von Neumann ρ-invariants

- M a closed 3-manifold, Γ a group
- Given $\phi : \pi_1(M) \to \Gamma$, whenever $(M, \phi) = \partial(W, \psi)$ for some compact oriented 4-manifold W and $\psi : \pi_1(W) \to \Gamma$

  $$\rho(M, \phi) = \sigma_\Gamma^{(2)}(W, \phi) - \sigma_0(W) \in \mathbb{R}$$

  where $\sigma_\Gamma^{(2)}$ is the von Neumann signature of the intersection form on $H_2(W; \mathbb{Z}\Gamma)$ and σ_0 is the ordinary signature.
- (Cheeger–Gromov 1985) ρ does not depend on W.
- (Cochran–Orr–Teichner 1999) If M_K is the zero surgery on a knot K in S^3, $\Gamma = \mathbb{Z}$ and $\phi \neq 1$, then $\rho(M_K, \phi)$ is the integral of the Levine–Tristram signature function.
- A proper choice of Γ gives a concordance invariant.

Knot Concordance

A knot K in S^3 is *slice* if $K = \partial D$ for a locally flat 2-disk D in B^4.

Two knots K_1 and K_2 are *concordant* if $K_1 \# - K_2$ is slice, where $-K$ denotes the mirror image of K with reversed orientation.

The knot concordance group $\mathcal{C}$:

> *Concordance is an equivalence relation and the concordance classes form an abelian group $\mathcal{C}$ under $\#$ operation.*

Cochran, Orr, Teichner (1999) found a filtration of $\mathcal{C}$:

$$\cdots \subset \mathcal{F}_{(n.5)} \subset \mathcal{F}_{(n)} \subset \cdots \subset \mathcal{F}_{(1.5)} \subset \mathcal{F}_{(1)} \subset \mathcal{F}_{(0.5)} \subset \mathcal{F}_{(0)} \subset \mathcal{C}$$

$\mathcal{F}_{(m)} =$ the set of m-solvable knots which is a subgroup of $\mathcal{C}$.

Higher order Blanchfield linking forms

COT in 1999 defined

- a series of groups Γ_n and (noncommutative) rings $\mathcal{R}_n$
- $\mathbb{Z}\Gamma_n \subset \mathcal{R}_n \subset \mathcal{K}_n$
- $\mathcal{K}_n$ denotes the (skew) field of quotients of $\mathbb{Z}\Gamma_n$
- $\Gamma_0 = \mathbb{Z} = \langle t \rangle$, $\mathcal{R}_0 = \mathbb{Q}[t^{\pm 1}]$, $\mathcal{K}_0 = \mathbb{Q}(t)$.

Consider

- $\mathcal{A}_0 = H_1(M_K; \mathcal{R}_0)$ the ordinary rational Alexander module
- $B\ell_0 : \mathcal{A}_0 \times \mathcal{A}_0 \to \mathcal{K}_0/\mathcal{R}_0$ the Blanchfield form

Higher order Blanchfield linking forms

COT proved

- An element $a_0 \in \mathcal{A}_0$ and $B\ell_0$ give a homomorphism $\phi_1 : \pi_1(M_K) \to \Gamma_1$ and an action of $\pi_1(M_K)$ on $\mathcal{R}_1$
- One can define $\mathcal{A}_1 = H_1(M_K; \mathcal{R}_1)$ together with a Blanchfield form $B\ell_1 : \mathcal{A}_1 \times \mathcal{A}_1 \to \mathcal{K}_1/\mathcal{R}_1$
- Iterating this procedure, given $(a_0, a_1, \ldots, a_{n-1})$ with $a_i \in \mathcal{A}_i$, one can define a homomorphism

$$\phi_n : \pi_1(M_K) \to \Gamma_n$$

and the n-th Alexander module

$$\mathcal{A}_n = H_1(M_K; \mathcal{R}_n)$$

together with the n-th Blanchfield form

$$B\ell_n : \mathcal{A}_n \times \mathcal{A}_n \to \mathcal{K}_n/\mathcal{R}_n$$

COT obstructions

Suppose K is $(n.5)$-solvable with $n \geq 1$.

- For $k \leq n$ and any $(a_0, \ldots, a_{k-1})$ with $a_i \in P_i$,
  $\mathcal{A}_k$ has a self-annihilating submodule P_k w.r.t. $B\ell_k$.
- There is $\phi_k = \phi(a_0, \ldots, a_{k-1}) : \pi_1(M_K) \to \Gamma_k$
  $\mathcal{A}_k$ and nonsingular $B\ell_k$ are defined
  $\rho(M_K, \phi_k)$ is defined and equal to zero.

We say that K has *vanishing COT obstructions of order n* if K satisfies these.

It is known that $H_1(M_{K\#J}; \mathcal{R}_k) = H_1(M_K, \mathcal{R}_k) \oplus H_1(M_J, \mathcal{R}_k)$.
However, no guarantee that $P_k^{K\#J} = P_k^K \oplus P_k^J$.

Splittings of ρ-invariants

Theorem. Suppose that $K\#J$ is slice (or $(n.5)$-solvable). (So, for $k \leq n$, there is $\phi(a_0, \ldots, a_{k-1}) : \pi(M_{K\#J}) \to \Gamma_k$ and a s.a.s $P_k^{K\#J} \subset H_1(M_{K\#J}; \mathcal{R}_k)$ and $\rho(M_{K\#J}, \phi_k) = 0$.)

Suppose, in addition, for each k, there is a s.a.s $P_k^K \subset H_1(M_K; \mathcal{R}_k)$ such that $P_k^K \subset P_k^{K\#J}$.

Then both K and J have vanishing COT obstructions of order n.

Known Splitting Properties

- (Levine 1969) Suppose that K_1 and K_2 have coprime Alexander polynomials. If $K_1 \# K_2$ is algebraically slice (Levine invariants vanish), so are both K_1 and K_2.

- (K 2001, 2007) Suppose that K_1 and K_2 have coprime Alexander polynomials. If $K_1 \# K_2$ have vanishing Casson-Gordon-Gilmer invariants, so do both K_1 and K_2.

- (K & T. Kim 2007) Suppose K_1 and K_2 have coprime Alexander polynomials. If $K_1 \# K_2$ has vanishing COT obstruction of order 1, so do both K_1 and K_2.

Remark. The additional condition in our theorem is a generalization: (Levine) If K and J have coprime Alexander polynomials, that there is a s.a.s P_0^K such that $P_0^K \subset P_0^{K\#J}$.

Proof of Theorem

1. $P_k^{K\#J} = P_k^K \oplus P_k^J$ since $B\ell_k$ is nonsingular.
2. Splitting property of ρ under connected sum:

$$\rho(M_{K\#J}, \phi) = \rho(M_K, \tilde{\phi}|_{\pi_1(M_K)}) + \rho(M_J, \tilde{\phi}|_{\pi_1(M_J)})$$

3. If a_i was chosen from P_i^K, then $\rho(M_J, \tilde{\phi}|_{\pi_1(M_J)}) = 0$.

Example

Let K be the knot given in COT paper.

K has nonvanishing COT obstruction of order 2.
$\mathcal{A}_0$ and $\mathcal{A}_1$ have unique self-annihilating submodules.
Alexander polynomial $(t^{-1} - 3 + t)^2$.

Let J be a knot with Alexander polynomial
$(lt - (l - 1))((l - 1) - lt)$ for $l \in \mathbb{Z}_+$.

Then K **and** J **are not concordant.**

Because:

$P_0^{K\#J} = P_0^K \oplus P_0^J$ because Alexander polynomials are coprime.
$\mathcal{A}_1$ has unique self-annihilating submodule
Using this, one can show $P_1^{K\#J} \cap H_1(M_K, \mathcal{R}_1) = P_1^K$.