Knot Theory Invariants in Algebraic Geometry

Laurentiu Maxim

University of Wisconsin-Madison

1. Motivation

2. Plane curve complements
   - Infinite cyclic invariants: *Alexander polynomials*
   - Universal abelian invariants: *Characteristic varieties*
   - L^2*-Betti numbers* and *Cochran-Harvey* invariants

3. Examples

- Serre's problem: Find restrictions imposed on a group by the fact that it can appear as fundamental group of a complex *algebraic* manifold.

- Serre's problem: Find restrictions imposed on a group by the fact that it can appear as fundamental group of a complex *algebraic* manifold.
- Most finitely presented groups cannot be *projective groups*, i.e., π_1 of a complex *projective* manifold. E.g., free abelian groups of *odd* rank (Hint: use Hodge theory).

- Serre's problem: Find restrictions imposed on a group by the fact that it can appear as fundamental group of a complex *algebraic* manifold.
- Most finitely presented groups cannot be *projective groups*, i.e., π_1 of a complex *projective* manifold. E.g., free abelian groups of *odd* rank (Hint: use Hodge theory).
- By contrast, Taubes (1992) showed that every finitely presented group is π_1 of a compact complex 3-manifold.

- Serre's problem: Find restrictions imposed on a group by the fact that it can appear as fundamental group of a complex *algebraic* manifold.
- Most finitely presented groups cannot be *projective groups*, i.e., π_1 of a complex *projective* manifold. E.g., free abelian groups of *odd* rank (Hint: use Hodge theory).
- By contrast, Taubes (1992) showed that every finitely presented group is π_1 of a compact complex 3-manifold.
- Morgan (1978), Kapovich-Milson (1997), etc. found infinitely many non-isomorphic examples of *non-quasiprojective groups*.

- Consider only complex algebraic manifolds which are complements to hypersurfaces in $\mathbb{C}^n$ (or in $\mathbb{CP}^n$).

- Consider only complex algebraic manifolds which are complements to hypersurfaces in $\mathbb{C}^n$ (or in $\mathbb{CP}^n$).
- Reduction to a low-dimensional topology problem: by a Zariski-Lefschetz type theorem, possible π_1's of complements to hypersurfaces in $\mathbb{C}^n$ are precisely the fundamental groups of complements to plane curves $\mathcal{C} = \{f(x, y) = 0\} \subset \mathbb{C}^2$.

- Consider only complex algebraic manifolds which are complements to hypersurfaces in $\mathbb{C}^n$ (or in $\mathbb{CP}^n$).
- Reduction to a low-dimensional topology problem: by a Zariski-Lefschetz type theorem, possible π_1's of complements to hypersurfaces in $\mathbb{C}^n$ are precisely the fundamental groups of complements to plane curves $\mathcal{C} = \{f(x, y) = 0\} \subset \mathbb{C}^2$.
- Question: What groups can be π_1 of complements to curves in $\mathbb{C}^2$? What obstruction are there? Similarly for $\mathbb{CP}^2$.

- Consider only complex algebraic manifolds which are complements to hypersurfaces in $\mathbb{C}^n$ (or in $\mathbb{CP}^n$).
- Reduction to a low-dimensional topology problem: by a Zariski-Lefschetz type theorem, possible π_1's of complements to hypersurfaces in $\mathbb{C}^n$ are precisely the fundamental groups of complements to plane curves $\mathcal{C} = \{f(x,y)=0\} \subset \mathbb{C}^2$.
- Question: What groups can be π_1 of complements to curves in $\mathbb{C}^2$? What obstruction are there? Similarly for $\mathbb{CP}^2$.
- E.g., many *knot groups* cannot be realized as $\pi_1(\mathbb{C}^2 - \mathcal{C})$ for a curve $\mathcal{C}$.

Setting

- Let $\mathcal{C} = \{f(x,y) = 0\}$ be a reduced curve in $\mathbb{C}^2$ of degree d.

Setting

- Let $\mathcal{C} = \{f(x,y) = 0\}$ be a reduced curve in $\mathbb{C}^2$ of degree d.
- Set $X = \mathbb{C}^2 - \mathcal{C}$, with $G = \pi_1(X)$.

Setting

- Let $\mathcal{C} = \{f(x,y) = 0\}$ be a reduced curve in $\mathbb{C}^2$ of degree d.
- Set $X = \mathbb{C}^2 - \mathcal{C}$, with $G = \pi_1(X)$.
- $H_1(G) = H_1(X) = \mathbb{Z}^s$, for $s = \#$ of irred. components of $\mathcal{C}$.

Setting

- Let $\mathcal{C} = \{f(x,y) = 0\}$ be a reduced curve in $\mathbb{C}^2$ of degree d.
- Set $X = \mathbb{C}^2 - \mathcal{C}$, with $G = \pi_1(X)$.
- $H_1(G) = H_1(X) = \mathbb{Z}^s$, for $s = \#$ of irred. components of $\mathcal{C}$.
- If the projective completion $\bar{\mathcal{C}}$ of $\mathcal{C}$ is *transversal* to the line at infinity, there is a central extension:

$$0 \to \mathbb{Z} \to G \to \pi_1(\mathbb{CP}^2 \setminus \bar{\mathcal{C}}) \to 0.$$

Setting

- Let $\mathcal{C} = \{f(x,y) = 0\}$ be a reduced curve in $\mathbb{C}^2$ of degree d.
- Set $X = \mathbb{C}^2 - \mathcal{C}$, with $G = \pi_1(X)$.
- $H_1(G) = H_1(X) = \mathbb{Z}^s$, for $s = \#$ of irred. components of $\mathcal{C}$.
- If the projective completion $\bar{\mathcal{C}}$ of $\mathcal{C}$ is *transversal* to the line at infinity, there is a central extension:

$$0 \to \mathbb{Z} \to G \to \pi_1(\mathbb{CP}^2 \setminus \bar{\mathcal{C}}) \to 0.$$

- Important obstructions on $G = \pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ and on the topology of $\mathcal{C}$ are derived by analyzing various invariants associated to covering spaces of the complement.

Infinite cyclic cover

- $lk : G = \pi_1(\mathbb{C}^2 - \mathcal{C}) \to \mathbb{Z}$, $\alpha \mapsto lk\#(\alpha, \mathcal{C})$.

Infinite cyclic cover

- $lk : G = \pi_1(\mathbb{C}^2 - \mathcal{C}) \to \mathbb{Z}$, $\alpha \mapsto lk\#(\alpha, \mathcal{C})$.
- $X^c :=$ covering of $X = \mathbb{C}^2 - \mathcal{C}$ corresponding to $Ker(lk)$.

Infinite cyclic cover

- $lk : G = \pi_1(\mathbb{C}^2 - \mathcal{C}) \to \mathbb{Z}$, $\alpha \mapsto lk\#(\alpha, \mathcal{C})$.
- $X^c :=$ covering of $X = \mathbb{C}^2 - \mathcal{C}$ corresponding to $Ker(lk)$.

Theorem (Zariski-Libgober)

$H_1(X^c; \mathbb{C})$ *is a torsion* $\mathbb{C}[t, t^{-1}]$*-module.*

Infinite cyclic cover

- $lk : G = \pi_1(\mathbb{C}^2 - \mathcal{C}) \to \mathbb{Z}$, $\alpha \mapsto lk\#(\alpha, \mathcal{C})$.
- $X^c :=$ covering of $X = \mathbb{C}^2 - \mathcal{C}$ corresponding to $Ker(lk)$.

Theorem (Zariski-Libgober)

$H_1(X^c; \mathbb{C})$ is a torsion $\mathbb{C}[t, t^{-1}]$-module.

Definition

$\Delta_{\mathcal{C}}(t) = \text{order} H_1(X^c; \mathbb{C})$ is the Alexander polynomial of $\mathcal{C}$ (or G).

Libgober's divisibility theorem for Alexander polynomials

- For each $x \in \mathrm{Sing}(\mathcal{C})$, let $L_x := S^3_x \cap \mathcal{C}$ be the link of x.

Libgober's divisibility theorem for Alexander polynomials

- For each $x \in \mathrm{Sing}(\mathcal{C})$, let $L_x := S^3_x \cap \mathcal{C}$ be the link of x.
- Milnor: There is a fibration $F_x \hookrightarrow S^3_x \setminus L_x \to S^1$

Libgober's divisibility theorem for Alexander polynomials

- For each $x \in \mathrm{Sing}(\mathcal{C})$, let $L_x := S^3_x \cap \mathcal{C}$ be the link of x.
- Milnor: There is a fibration $F_x \hookrightarrow S^3_x \setminus L_x \to S^1$
- The Milnor fibre F_x is homotopy equivalent to a join of circles, their number being equal to the Milnor number $\mu(\mathcal{C}, x)$.

Libgober's divisibility theorem for Alexander polynomials

- For each $x \in \mathrm{Sing}(\mathcal{C})$, let $L_x := S^3_x \cap \mathcal{C}$ be the link of x.
- Milnor: There is a fibration $F_x \hookrightarrow S^3_x \setminus L_x \to S^1$
- The Milnor fibre F_x is homotopy equivalent to a join of circles, their number being equal to the Milnor number $\mu(\mathcal{C}, x)$.
- Let $h_x : F_x \to F_x$ be the monodromy homeomorphism

Libgober's divisibility theorem for Alexander polynomials

- For each $x \in \mathrm{Sing}(\mathcal{C})$, let $L_x := S^3_x \cap \mathcal{C}$ be the link of x.
- Milnor: There is a fibration $F_x \hookrightarrow S^3_x \setminus L_x \to S^1$
- The Milnor fibre F_x is homotopy equivalent to a join of circles, their number being equal to the Milnor number $\mu(\mathcal{C}, x)$.
- Let $h_x : F_x \to F_x$ be the monodromy homeomorphism
- The local Alexander polynomial at x is defined by

$$\Delta_x(t) := \det\left(tI - (h_x)_* : H_1(F_x) \to H_1(F_x)\right)$$

Theorem (Libgober)

$\Delta_{\mathcal{C}}(t)$ *divides (up to a power of* $(t-1)$*)* $\prod_{x \in Sing(\mathcal{C})} \Delta_x(t)$

Theorem (Libgober)

$\Delta_{\mathcal{C}}(t)$ *divides (up to a power of* $(t-1)$*)* $\prod_{x \in Sing(\mathcal{C})} \Delta_x(t)$

- So the local type of singularities affects the topology of $\mathcal{C}$.

Theorem (Libgober)

$\Delta_{\mathcal{C}}(t)$ divides (up to a power of $(t-1)$) $\prod_{x \in Sing(\mathcal{C})} \Delta_x(t)$

- So the local type of singularities affects the topology of $\mathcal{C}$.
- Zariski showed that the position of singularities has effect on the topology of $\mathcal{C}$.

Theorem (Libgober)

$\Delta_{\mathcal{C}}(t)$ *divides (up to a power of* $(t-1)$*)* $\prod_{x \in Sing(\mathcal{C})} \Delta_x(t)$

- So the local type of singularities affects the topology of $\mathcal{C}$.
- Zariski showed that the position of singularities has effect on the topology of $\mathcal{C}$.
- Moreover, $\Delta_{\mathcal{C}}(t)$ is sensitive to the position of singularities (Libgober).

Corollary

- *The zeros of* $\Delta_{\mathcal{C}}(t)$ *are roots of unity.*

Corollary

- *The zeros of $\Delta_{\mathcal{C}}(t)$ are roots of unity.*
- *Moreover, for a curve $\mathcal{C}$ in general position at infinity, the zeros of $\Delta_{\mathcal{C}}(t)$ are roots of unity of order $d = deg(\mathcal{C})$.*

Corollary

- *The zeros of $\Delta_{\mathcal{C}}(t)$ are roots of unity.*
- *Moreover, for a curve $\mathcal{C}$ in general position at infinity, the zeros of $\Delta_{\mathcal{C}}(t)$ are roots of unity of order $d = deg(\mathcal{C})$.*

Corollary

- *The zeros of $\Delta_{\mathcal{C}}(t)$ are roots of unity.*
- *Moreover, for a curve $\mathcal{C}$ in general position at infinity, the zeros of $\Delta_{\mathcal{C}}(t)$ are roots of unity of order $d = deg(\mathcal{C})$.*

Example

- Many knot groups, e.g. that of figure eight knot (whose Alexander polynomial is $t^2 - 3t + 1$), cannot be of the form $\pi_1(\mathbb{C}^2 - \mathcal{C})$.

Corollary

- *The zeros of $\Delta_{\mathcal{C}}(t)$ are roots of unity.*
- *Moreover, for a curve $\mathcal{C}$ in general position at infinity, the zeros of $\Delta_{\mathcal{C}}(t)$ are roots of unity of order $d = deg(\mathcal{C})$.*

Example

- Many knot groups, e.g. that of figure eight knot (whose Alexander polynomial is $t^2 - 3t + 1$), cannot be of the form $\pi_1(\mathbb{C}^2 - \mathcal{C})$.
- However, the class of possible π_1 of plane curve complements includes braid groups, or groups of torus knots of type (p, q).

- The Alexander polynomial doesn't provide enough information for reducible curves (with $s \geq 2$ irreducible components).

- The Alexander polynomial doesn't provide enough information for reducible curves (with $s \geq 2$ irreducible components).
- E.g., M. Oka showed that if $\mathcal{C}$ is a union of two curves that intersect transversally, then $\Delta_{\mathcal{C}}(t) = (t-1)^{s-1}$.

- The Alexander polynomial doesn't provide enough information for reducible curves (with $s \geq 2$ irreducible components).
- E.g., M. Oka showed that if $\mathcal{C}$ is a union of two curves that intersect transversally, then $\Delta_{\mathcal{C}}(t) = (t-1)^{s-1}$.
- To overcome this problem, we study higher coverings of the complement, or more sophisticated invariants (e.g., L^2-Betti numbers) of the infinite cyclic covering.

Universal abelian cover

- Let X^{ab} be the cover of $X = \mathbb{C}^2 - \mathcal{C}$ corresponding to $G' = [G, G]$.

Universal abelian cover

- Let X^{ab} be the cover of $X = \mathbb{C}^2 - \mathcal{C}$ corresponding to $G' = [G, G]$.
- The universal abelian module

$$\mathcal{A}^{\mathbb{C}} := H_1(X^{ab}; \mathbb{C}) = G'/G'' \otimes \mathbb{C}$$

is finitely generated over $\mathbb{C}[G/G'] = \mathbb{C}[t_1^{\pm 1}, .., t_s^{\pm 1}] =: R_s$

Universal abelian cover

- Let X^{ab} be the cover of $X = \mathbb{C}^2 - \mathcal{C}$ corresponding to $G' = [G, G]$.
- The universal abelian module

$$\mathcal{A}^{\mathbb{C}} := H_1(X^{ab}; \mathbb{C}) = G'/G'' \otimes \mathbb{C}$$

is finitely generated over $\mathbb{C}[G/G'] = \mathbb{C}[t_1^{\pm 1}, .., t_s^{\pm 1}] =: R_s$

- The support $\mathrm{Supp}(\mathcal{A}^{\mathbb{C}})$ of $\mathcal{A}^{\mathbb{C}}$ is the sub-scheme of the s-dim. torus $(\mathbb{C}^*)^s = \mathrm{Spec}(R_s)$ defined by the order ideal of $\mathcal{A}^{\mathbb{C}}$.

Universal abelian cover

- Let X^{ab} be the cover of $X = \mathbb{C}^2 - \mathcal{C}$ corresponding to $G' = [G, G]$.
- The universal abelian module

$$\mathcal{A}^{\mathbb{C}} := H_1(X^{ab}; \mathbb{C}) = G'/G'' \otimes \mathbb{C}$$

is finitely generated over $\mathbb{C}[G/G'] = \mathbb{C}[t_1^{\pm 1}, .., t_s^{\pm 1}] =: R_s$

- The support $\mathrm{Supp}(\mathcal{A}^{\mathbb{C}})$ of $\mathcal{A}^{\mathbb{C}}$ is the sub-scheme of the s-dim. torus $(\mathbb{C}^*)^s = \mathrm{Spec}(R_s)$ defined by the order ideal of $\mathcal{A}^{\mathbb{C}}$.

Example

Universal abelian cover

- Let X^{ab} be the cover of $X = \mathbb{C}^2 - \mathcal{C}$ corresponding to $G' = [G, G]$.
- The universal abelian module

$$\mathcal{A}^{\mathbb{C}} := H_1(X^{ab}; \mathbb{C}) = G'/G'' \otimes \mathbb{C}$$

is finitely generated over $\mathbb{C}[G/G'] = \mathbb{C}[t_1^{\pm 1}, .., t_s^{\pm 1}] =: R_s$

- The support $\mathrm{Supp}(\mathcal{A}^{\mathbb{C}})$ of $\mathcal{A}^{\mathbb{C}}$ is the sub-scheme of the s-dim. torus $(\mathbb{C}^*)^s = \mathrm{Spec}(R_s)$ defined by the order ideal of $\mathcal{A}^{\mathbb{C}}$.

Example

- If $\mathcal{C}$ is irreducible, then $\mathrm{Supp}(\mathcal{A}^{\mathbb{C}}) = \{\Delta_{\mathcal{C}}(t) = 0\}$.

Universal abelian cover

- Let X^{ab} be the cover of $X = \mathbb{C}^2 - \mathcal{C}$ corresponding to $G' = [G, G]$.
- The universal abelian module

$$\mathcal{A}^{\mathbb{C}} := H_1(X^{ab}; \mathbb{C}) = G'/G'' \otimes \mathbb{C}$$

is finitely generated over $\mathbb{C}[G/G'] = \mathbb{C}[t_1^{\pm 1}, .., t_s^{\pm 1}] =: R_s$

- The support $\mathrm{Supp}(\mathcal{A}^{\mathbb{C}})$ of $\mathcal{A}^{\mathbb{C}}$ is the sub-scheme of the s-dim. torus $(\mathbb{C}^*)^s = \mathrm{Spec}(R_s)$ defined by the order ideal of $\mathcal{A}^{\mathbb{C}}$.

Example

- If $\mathcal{C}$ is irreducible, then $\mathrm{Supp}(\mathcal{A}^{\mathbb{C}}) = \{\Delta_{\mathcal{C}}(t) = 0\}$.
- If L is a link in S^3 with $G = \pi_1(S^3 \setminus L)$, then $\mathrm{Supp}(\mathcal{A}^{\mathbb{C}})$ is the zero-set of the multivariable Alexander polynomial of L.

Theorem (Libgober)

If $\mathcal{C}$ is a curve in general position at infinity, then

$$\mathrm{Supp}(\mathcal{A}^{\mathbb{C}}) \subset \{(\lambda_1, \cdots, \lambda_s) \in (\mathbb{C}^*)^s \mid \prod_{i=1}^{s} \lambda_i^{d_i} = 1\}.$$

Theorem (Libgober)

If $\mathcal{C}$ is a curve in general position at infinity, then

$$\mathrm{Supp}(\mathcal{A}^{\mathbb{C}}) \subset \{(\lambda_1, \cdots, \lambda_s) \in (\mathbb{C}^*)^s \mid \prod_{i=1}^{s} \lambda_i^{d_i} = 1\}.$$

Theorem (Arapura)

$\mathrm{Supp}(\mathcal{A}^{\mathbb{C}})$ *is a union of translated subtori of* $(\mathbb{C}^*)^s$.

Theorem (Libgober)

If $\mathcal{C}$ is a curve in general position at infinity, then

$$\mathrm{Supp}(\mathcal{A}^{\mathbb{C}}) \subset \{(\lambda_1, \cdots, \lambda_s) \in (\mathbb{C}^*)^s \mid \prod_{i=1}^{s} \lambda_i^{d_i} = 1\}.$$

Theorem (Arapura)

$\mathrm{Supp}(\mathcal{A}^{\mathbb{C}})$ *is a union of translated subtori of* $(\mathbb{C}^*)^s$.

Remark (Libgober)

These subtori can be identified in terms of local type of singularities and the configuration of singular points.

L^2-Betti numbers

Studying higher (solvable) covers of $X = \mathbb{C}^2 \setminus \mathcal{C}$, i.e., associated to higher terms in the derived series of $G = \pi_1(X)$, amounts to considering certain L^2-Betti numbers of the infinite cyclic cover X^c.

To any space X and group homomorphism $\alpha : \pi_1(X) \to \Gamma$, we associate L^2-Betti numbers

$$b_p^{(2)}(X, \alpha) := \dim_{\mathcal{N}(\Gamma)} H_p\left(C_*(X_\alpha) \otimes_{\mathbb{Z}\Gamma} \mathcal{N}(\Gamma)\right) \in [0, \infty],$$

where X_α is the covering of X defined by α, and $\mathcal{N}(\Gamma)$ is the von Neumann algebra of Γ, so that:

To any space X and group homomorphism $\alpha : \pi_1(X) \to \Gamma$, we associate L^2-Betti numbers

$$b_p^{(2)}(X, \alpha) := \dim_{\mathcal{N}(\Gamma)} H_p\left(C_*(X_\alpha) \otimes_{\mathbb{Z}\Gamma} \mathcal{N}(\Gamma)\right) \in [0, \infty],$$

where X_α is the covering of X defined by α, and $\mathcal{N}(\Gamma)$ is the von Neumann algebra of Γ, so that:

- $b_p^{(2)}(X, \alpha)$ is a homotopy invariant of the pair (X, α).

To any space X and group homomorphism $\alpha : \pi_1(X) \to \Gamma$, we associate L^2-Betti numbers

$$b_p^{(2)}(X, \alpha) := \dim_{\mathcal{N}(\Gamma)} H_p \left(C_*(X_\alpha) \otimes_{\mathbb{Z}\Gamma} \mathcal{N}(\Gamma) \right) \in [0, \infty],$$

where X_α is the covering of X defined by α, and $\mathcal{N}(\Gamma)$ is the von Neumann algebra of Γ, so that:

- $b_p^{(2)}(X, \alpha)$ is a homotopy invariant of the pair (X, α).
- if X is a finite CW-complex,

$$\sum_p (-1)^p b_p^{(2)}(X, \alpha) = \chi(X) = \sum_p (-1)^p \beta_p(X)$$

To any space X and group homomorphism $\alpha : \pi_1(X) \to \Gamma$, we associate L^2-Betti numbers

$$b_p^{(2)}(X, \alpha) := \dim_{\mathcal{N}(\Gamma)} H_p\left(C_*(X_\alpha) \otimes_{\mathbb{Z}\Gamma} \mathcal{N}(\Gamma)\right) \in [0, \infty],$$

where X_α is the covering of X defined by α, and $\mathcal{N}(\Gamma)$ is the von Neumann algebra of Γ, so that:

- $b_p^{(2)}(X, \alpha)$ is a homotopy invariant of the pair (X, α).
- if X is a finite CW-complex,

$$\sum_p (-1)^p b_p^{(2)}(X, \alpha) = \chi(X) = \sum_p (-1)^p \beta_p(X)$$

- if $\mathrm{Image}(\alpha) \subset \bar{\Gamma} \subset \Gamma$, then

$$b_p^{(2)}(X, \alpha : \pi_1(X) \to \bar{\Gamma}) = b_p^{(2)}(X, \alpha : \pi_1(X) \to \Gamma)$$

Back to Curves

- $\alpha : \pi_1(X) \to \Gamma$ is called admissible if $lk : \pi_1(X) \to \mathbb{Z}$ factors through α.

Back to Curves

- $\alpha : \pi_1(X) \to \Gamma$ is called admissible if $lk : \pi_1(X) \to \mathbb{Z}$ factors through α.
- For admissible α, let $\bar{\Gamma} := \mathrm{Im}(\pi_1(X^c) \to \pi_1(X) \xrightarrow{\alpha} \Gamma)$, and $\bar{\alpha} : \pi_1(X^c) \to \bar{\Gamma}$ the induced map.

Back to Curves

- $\alpha : \pi_1(X) \to \Gamma$ is called admissible if $lk : \pi_1(X) \to \mathbb{Z}$ factors through α.
- For admissible α, let $\bar{\Gamma} := \mathrm{Im}(\pi_1(X^c) \to \pi_1(X) \xrightarrow{\alpha} \Gamma)$, and $\bar{\alpha} : \pi_1(X^c) \to \bar{\Gamma}$ the induced map.
- Consider $b_p^{(2)}(X, \alpha)$, $p \geq 0$, and $b_1^{(2)}(X^c, \bar{\alpha})$.

Back to Curves

- $\alpha : \pi_1(X) \to \Gamma$ is called admissible if $lk : \pi_1(X) \to \mathbb{Z}$ factors through α.
- For admissible α, let $\bar{\Gamma} := \mathrm{Im}(\pi_1(X^c) \to \pi_1(X) \xrightarrow{\alpha} \Gamma)$, and $\bar{\alpha} : \pi_1(X^c) \to \bar{\Gamma}$ the induced map.
- Consider $b_p^{(2)}(X, \alpha)$, $p \geq 0$, and $b_1^{(2)}(X^c, \bar{\alpha})$.
- A priori, there is no reason to expect $b_1^{(2)}(X^c, \bar{\alpha})$ to be finite (as X^c is an infinite CW complex).

Theorem A (Friedl-Leidy-M.)

If $X = \mathbb{C}^2 - \mathcal{C}$ *for some curve* $\mathcal{C}$ *in general position at infinity, and* $\alpha : \pi_1(X) \to \Gamma$ *is admissible, then*

$$b_p^{(2)}(X, \alpha) = \begin{cases} 0, & p \neq 2, \\ \chi(X), & p = 2. \end{cases}$$

Theorem A (Friedl-Leidy-M.)

If $X = \mathbb{C}^2 - \mathcal{C}$ for some curve $\mathcal{C}$ in general position at infinity, and $\alpha : \pi_1(X) \to \Gamma$ is admissible, then

$$b_p^{(2)}(X, \alpha) = \begin{cases} 0, & p \neq 2, \\ \chi(X), & p = 2. \end{cases}$$

Corollary

$b_p^{(2)}(X, \alpha)$ ($p \geq 0$) depends only on the degree of $\mathcal{C}$ and on the local type of singularities, and is independent on α and on the position of singularities of $\mathcal{C}$. Indeed,

$$b_2^{(2)}(X, \alpha) = (d-1)^2 - \sum_{x \in \mathrm{Sing}(\mathcal{C})} \mu(\mathcal{C}, x).$$

Obstructions on the L^2-Betti numbers of curves

Theorem B (Friedl-Leidy-M.)

If $X = \mathbb{C}^2 - \mathcal{C}$ for some curve $\mathcal{C}$ in general position at infinity, then $b_1^{(2)}(X^c, \bar{\alpha})$ is finite, and an upper bound is determined by the local type of singularities of $\mathcal{C}$.

Obstructions on the L^2-Betti numbers of curves

Theorem B (Friedl-Leidy-M.)

If $X = \mathbb{C}^2 - \mathcal{C}$ for some curve $\mathcal{C}$ in general position at infinity, then $b_1^{(2)}(X^c, \bar{\alpha})$ is finite, and an upper bound is determined by the local type of singularities of $\mathcal{C}$. More precisely,

$$b_1^{(2)}(X^c, \bar{\alpha}) \leq \sum_{x \in \mathrm{Sing}(\mathcal{C})} (\mu(\mathcal{C}, x) + n_x - 1) + 2g + d,$$

where n_x is the number of branches through $x \in \mathrm{Sing}(\mathcal{C})$ and g is the genus of the normalization of $\mathcal{C}$.

Remark

$b_1^{(2)}(X^c, \bar{\alpha})$ depends in general on the position of singularities of $\mathcal{C}$.

Relation with Cochran-Harvey invariants

Definition (Rational derived series of a group)

- Let $G_r^{(0)} = G$.

Relation with Cochran-Harvey invariants

Definition (Rational derived series of a group)

- Let $G_r^{(0)} = G$.
- For $n \geq 1$, the n^{th} term of the rational derived series of G is:

$$G_r^{(n)} = \{g \in G_r^{(n-1)} | g^k \in [G_r^{(n-1)}, G_r^{(n-1)}], \text{ for some } k \in \mathbb{Z} - 0\}$$

Relation with Cochran-Harvey invariants

Definition (Rational derived series of a group)

- Let $G_r^{(0)} = G$.
- For $n \geq 1$, the n^{th} term of the rational derived series of G is:

$$G_r^{(n)} = \{g \in G_r^{(n-1)} | g^k \in [G_r^{(n-1)}, G_r^{(n-1)}], \text{ for some } k \in \mathbb{Z}-0\}$$

- $G_r^{(i)} \triangleleft G_r^{(j)} \triangleleft G$, if $i \geq j \geq 0$.

Relation with Cochran-Harvey invariants

Definition (Rational derived series of a group)

- Let $G_r^{(0)} = G$.
- For $n \geq 1$, the n^{th} term of the rational derived series of G is:

$$G_r^{(n)} = \{g \in G_r^{(n-1)} | g^k \in [G_r^{(n-1)}, G_r^{(n-1)}], \text{ for some } k \in \mathbb{Z} - 0\}$$

- $G_r^{(i)} \triangleleft G_r^{(j)} \triangleleft G$, if $i \geq j \geq 0$.
- Set $\Gamma_n := G/G_r^{(n+1)}$ and $\alpha_n : G \to \Gamma_n$ the induced map.

- Let $\bar{\Gamma}_n := \mathrm{Im}\{\pi_1(X^c) \to \pi_1(X) \xrightarrow{\alpha_n} \Gamma_n\}$ and $\bar{\alpha}_n : \pi_1(X^c) \to \bar{\Gamma}_n$ the induced map.

- Let $\bar{\Gamma}_n := \mathrm{Im}\{\pi_1(X^c) \to \pi_1(X) \xrightarrow{\alpha_n} \Gamma_n\}$ and $\bar{\alpha}_n : \pi_1(X^c) \to \bar{\Gamma}_n$ the induced map.
- Set $\delta_n(\mathcal{C}) := b_1^{(2)}(X^c, \bar{\alpha}_n)$.

- Let $\bar{\Gamma}_n := \mathrm{Im}\{\pi_1(X^c) \to \pi_1(X) \xrightarrow{\alpha_n} \Gamma_n\}$ and $\bar{\alpha}_n : \pi_1(X^c) \to \bar{\Gamma}_n$ the induced map.
- Set $\delta_n(\mathcal{C}) := b_1^{(2)}(X^c, \bar{\alpha}_n)$.

- Let $\bar{\Gamma}_n := \mathrm{Im}\{\pi_1(X^c) \to \pi_1(X) \xrightarrow{\alpha_n} \Gamma_n\}$ and $\bar{\alpha}_n : \pi_1(X^c) \to \bar{\Gamma}_n$ the induced map.
- Set $\delta_n(\mathcal{C}) := b_1^{(2)}(X^c, \bar{\alpha_n})$.

Theorem (Friedl-Leidy-M.)

$\delta_n(\mathcal{C})$ coincides with the Cochran-Harvey n^{th} order degree of $G = \pi_1(X)$.

- Let $\bar{\Gamma}_n := \mathrm{Im}\{\pi_1(X^c) \to \pi_1(X) \xrightarrow{\alpha_n} \Gamma_n\}$ and $\bar{\alpha}_n : \pi_1(X^c) \to \bar{\Gamma}_n$ the induced map.
- Set $\delta_n(\mathcal{C}) := b_1^{(2)}(X^c, \bar{\alpha}_n)$.

Theorem (Friedl-Leidy-M.)

$\delta_n(\mathcal{C})$ coincides with the Cochran-Harvey n^{th} order degree of $G = \pi_1(X)$.

- So $\delta_n(\mathcal{C})$ is an integral invariant of G measuring the "size" of $G_r^{(n+1)}/G_r^{(n+2)}$ in the same way the degree of $\Delta_{\mathcal{C}}(t)$ measures the size of the infinite cyclic Alexander module.

- Let $\bar{\Gamma}_n := \mathrm{Im}\{\pi_1(X^c) \to \pi_1(X) \xrightarrow{\alpha_n} \Gamma_n\}$ and $\bar{\alpha}_n : \pi_1(X^c) \to \bar{\Gamma}_n$ the induced map.
- Set $\delta_n(\mathcal{C}) := b_1^{(2)}(X^c, \bar{\alpha}_n)$.

Theorem (Friedl-Leidy-M.)

$\delta_n(\mathcal{C})$ coincides with the Cochran-Harvey n^{th} order degree of $G = \pi_1(X)$.

- So $\delta_n(\mathcal{C})$ is an integral invariant of G measuring the "size" of $G_r^{(n+1)}/G_r^{(n+2)}$ in the same way the degree of $\Delta_{\mathcal{C}}(t)$ measures the size of the infinite cyclic Alexander module.
- In fact, if $\mathcal{C}$ is irreducible, $\delta_0(\mathcal{C}) = \deg \Delta_{\mathcal{C}}(t)$.

- Consequences of finiteness property: Free groups $\mathbb{F}_m$ with $m \geq 2$ cannot be of the form $\pi_1(\mathbb{C}^2 - \mathcal{C})$, for $\mathcal{C}$ a curve in general position at infinity, and similarly for groups of boundary links (those links whose components admit mutually disjoint Seifert surfaces). Indeed, some δ_n is infinite for these groups (cf. Harvey).

- The degrees $\delta_n(\mathcal{C})$ may be computed by means of Fox free calculus by using a presentation of $\pi_1(\mathbb{C}^2 - \mathcal{C})$.

- The degrees $\delta_n(\mathcal{C})$ may be computed by means of Fox free calculus by using a presentation of $\pi_1(\mathbb{C}^2 - \mathcal{C})$.
- A presentation can be obtained by means of Moishezon's braid monodromy.

Examples

- If $\mathcal{C}$ is defined by a weighted homogeneous polynomial $f(x,y)=0$ in $\mathbb{C}^2$, then G is the group of an (algebraic) fibered link, and

Examples

- If $\mathcal{C}$ is defined by a weighted homogeneous polynomial $f(x, y) = 0$ in $\mathbb{C}^2$, then G is the group of an (algebraic) fibered link, and
  - if either $n > 0$ or $\beta_1(X) > 1$, then $\delta_n(\mathcal{C}) = \mu(\mathcal{C}, 0) - 1$,

Examples

- If $\mathcal{C}$ is defined by a weighted homogeneous polynomial $f(x,y)=0$ in $\mathbb{C}^2$, then G is the group of an (algebraic) fibered link, and
  - if either $n>0$ or $\beta_1(X)>1$, then $\delta_n(\mathcal{C})=\mu(\mathcal{C},0)-1$,
  - if $\beta_1(X)=1$, then $\delta_0(\mathcal{C})=\mu(\mathcal{C},0)$, where $\mu(\mathcal{C},0)$ is the Milnor number of the singularity germ at the origin.

Examples

- If $\mathcal{C}$ is defined by a weighted homogeneous polynomial $f(x,y)=0$ in $\mathbb{C}^2$, then G is the group of an (algebraic) fibered link, and
  - if either $n>0$ or $\beta_1(X)>1$, then $\delta_n(\mathcal{C})=\mu(\mathcal{C},0)-1$,
  - if $\beta_1(X)=1$, then $\delta_0(\mathcal{C})=\mu(\mathcal{C},0)$, where $\mu(\mathcal{C},0)$ is the Milnor number of the singularity germ at the origin.
- If $\beta_1(G)=1$ then $\delta_0(\mathcal{C})=deg\Delta_{\mathcal{C}}(t)$. Moreover, in this case, if $\delta_0=0$ then $\delta_n=0$ for all n.

Example

Let $\bar{\mathcal{C}} \subset \mathbb{CP}^2$ be an irreducible degree d curve having *only nodes and cusps* as its only singularities.
Set $\mathcal{C} := \bar{\mathcal{C}} - L_\infty$, for L_∞ a generic line at infinity in $\mathbb{CP}^2$.

Example

Let $\bar{\mathcal{C}} \subset \mathbb{CP}^2$ be an irreducible degree d curve having *only nodes and cusps* as its only singularities.
Set $\mathcal{C} := \bar{\mathcal{C}} - L_\infty$, for L_∞ a generic line at infinity in $\mathbb{CP}^2$.
If $d \not\equiv 0 \pmod 6$, then $\delta_n(\mathcal{C}) = 0$ for all n. (this follows from the divisibility results on $\Delta_{\mathcal{C}}(t)$, which imply $\Delta_{\mathcal{C}}(t) = 1$).

Zariski's sextics with 6 cusps

Example

Let $\bar{\mathcal{C}} \subset \mathbb{CP}^2$ be a curve of degree 6 with only 6 cusps.
Set $\mathcal{C} := \bar{\mathcal{C}} - L_\infty$, for L_∞ a generic line at infinity in $\mathbb{CP}^2$.

Zariski's sextics with 6 cusps

Example

Let $\bar{\mathcal{C}} \subset \mathbb{CP}^2$ be a curve of degree 6 with only 6 cusps.
Set $\mathcal{C} := \bar{\mathcal{C}} - L_\infty$, for L_∞ a generic line at infinity in $\mathbb{CP}^2$.

- If the 6 cusps are on a conic, then $\pi_1(\mathbb{C}^2 - \mathcal{C})$ is isomorphic to π_1 of the trefoil knot, and has Alexander polynomial $t^2 - t + 1$. Thus $\delta_0(\mathcal{C}) = 2$, and $\delta_n(\mathcal{C}) = 1$ for all $n > 0$.

Zariski's sextics with 6 cusps

Example

Let $\bar{\mathcal{C}} \subset \mathbb{CP}^2$ be a curve of degree 6 with only 6 cusps.
Set $\mathcal{C} := \bar{\mathcal{C}} - L_\infty$, for L_∞ a generic line at infinity in $\mathbb{CP}^2$.

- If the 6 cusps are on a conic, then $\pi_1(\mathbb{C}^2 - \mathcal{C})$ is isomorphic to π_1 of the trefoil knot, and has Alexander polynomial $t^2 - t + 1$. Thus $\delta_0(\mathcal{C}) = 2$, and $\delta_n(\mathcal{C}) = 1$ for all $n > 0$.
- If the six cusps are not on a conic, then $\pi_1(\mathbb{C}^2 - \mathcal{C})$ is abelian. Therefore, $\delta_n(\mathcal{C}) = 0$ for all $n \geq 0$.

Corollary

The L^2-Betti numbers $\delta_n(\mathcal{C})$ are sensitive to the position of singular points of $\mathcal{C}$.

Corollary

The L^2-Betti numbers $\delta_n(\mathcal{C})$ are sensitive to the position of singular points of $\mathcal{C}$.

Open Problem

Find examples of Zariski pairs that are distinguished only by some L^2-Betti numbers of the infinite cyclic cover of the complement.

THANK YOU !!!