The $A - B$ slice problem and topological arbiters

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History and motivation:

Geometric classification tools in higher dimensions:

Surgery: Given an $n-$dimensional Poincaré complex X, is there an $n-$manifold M^n homotopy equivalent to it?

s-cobordism theorem: Given an $(n+1)$-dimensional s-cobordism W with $\partial W = M_1 \sqcup (-M_2)$, is W isomorphic to the product $M_1 \times [0,1]$?

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Both statements make sense for a fixed fundamental group.

In dimension $n = 4$: *smoothly* both surgery and s-cobordism fail even in the simply-connected case (Donaldson)

Dimension $n = 4$, topological category:

Freedman (1982): Both surgery and s-cobordism conjectures hold for $\pi_1 = 1$ and more generally for elementary amenable groups.

Applications:

- Classification of topological simply-connected $4-$manifolds.
- Slice results for knots and links, in particular: Alexander polynomial 1 knots are slice.
- (Quinn): Classification of homeomorphisms (up to isotopy) of simply-connected $4-$manifolds.

Conjecture (Freedman 1983) Surgery fails for free groups.

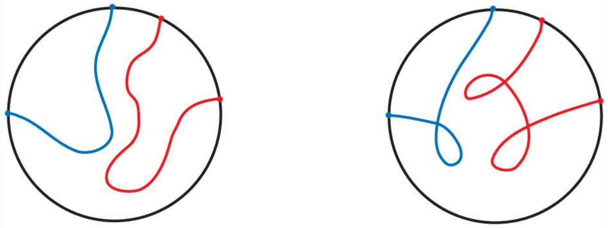
More specifically, there does not exist a topological $4-$manifold M, homotopy equivalent to $\vee^3 S^1$, with $\partial M = \mathcal{S}_0(Wh(Bor))$.

Equivalently: The Whitehead double of the Borromean rings is not a "free" slice link.

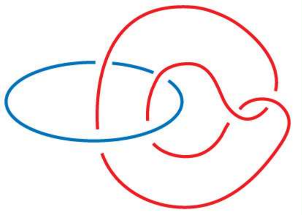
Link homotopy versus link concordance:

A link $L = (l_1, \ldots, l_n)$ is slice if its components bound disjoint embedded disks in D^4.

L is null-homotopic if its components bound disjoint disks in D^4 that may have self-intersections.



The Whitehead link: null-homotopic, not slice.



The effect of a crossing change on the fundamental group of the complement: relation $[m_i^g, m_i^h] = 1$.

The Milnor group of a link $L = (l_1, \ldots, l_n)$:

$$ML := \pi_1(S^3 \smallsetminus L) / \ll [m_i^g, m_i^h], g, h \in \pi_1(S^3 \smallsetminus L), i = 1, \ldots, n \gg$$

Milnor: A link $L = (l_1, \ldots, l_n)$ is null-homotopic if and only if $ML \cong M\,Free_{m_1,\ldots,m_n}$.

The Milnor group is nilpotent of class $n =$number of link components.

Algebraic manipulations in the Milnor group $\leftrightarrow$ geometric operations on the link.

A *decomposition* of D^4, $D^4 = A \cup B$, is an extension to the $4-$ball of the standard genus one Heegaard decomposition of the $3-$sphere. Specified distinguished curves $\alpha \subset \partial A, \beta \subset \partial B$ form the Hopf link in $S^3 = \partial D^4$.

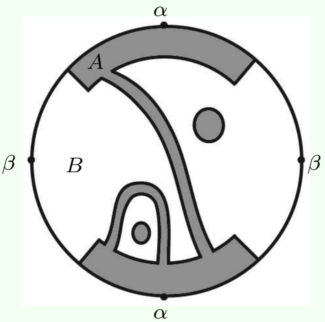
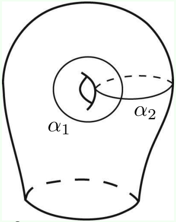
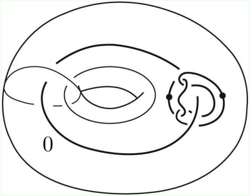


Figure: A $2-$dimensional example of a decomposition, $D^2 = A \cup B$.

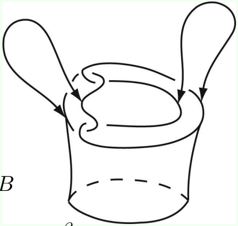
A



α



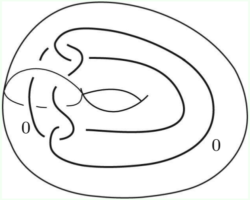
H_1



H_2

B

β



Iterate to get examples of *model decompositions* (introduced by M. Freedman and X.-S. Lin):

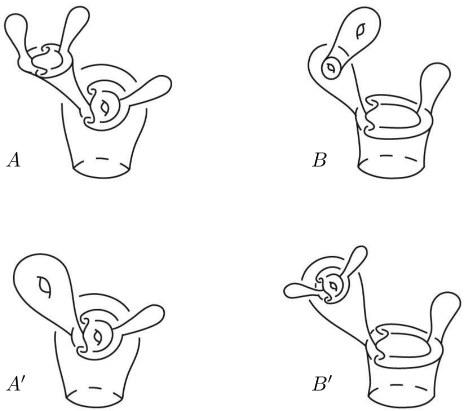


Figure: Examples of model decompositions of height 3.

An $n-$component link $L \subset S^3$ is *weakly A-B slice* if there exist decompositions $(A_i, B_i), i = 1, \ldots, n$ of D^4 and disjoint embeddings of all $2n$ manifolds $\{A_i, B_i\}$ into D^4 so that the distinguished curves $(\alpha_1, \ldots, \alpha_n)$ form the link L, and the curves $(\beta_1, \ldots, \beta_n)$ form a parallel copy of L.

L is A-B slice if, in addition, the new embeddings $A_i \subset D^4, B_i \subset D^4$ are standard: isotopic to the original embeddings.

Conjecture (M.Freedman 1983) 4-dimensional topological surgery fails for free groups.

More specifically, there does not exist a topological $4-$manifold M, homotopy equivalent to $\vee^3 S^1$, with $\partial M = \mathcal{S}_0(Wh(Bor))$.

Equivalently: The Whitehead double of the Borromean rings is not a "free" slice link.

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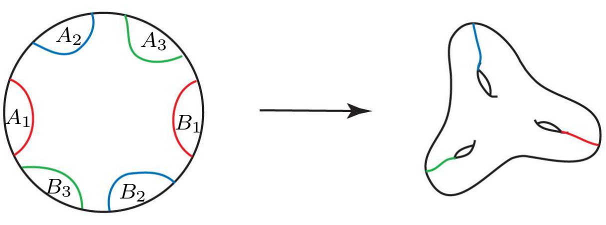
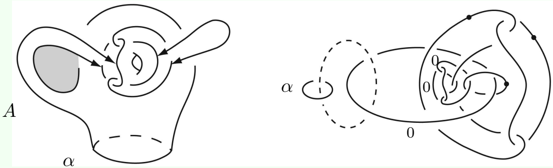
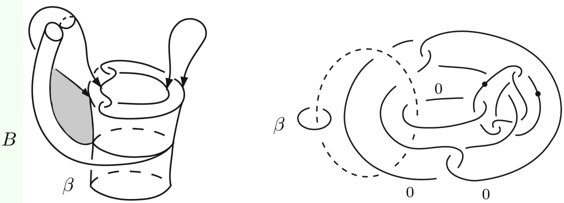
Connection with the surgery conjecture:

Suppose the existence of M^4, homotopy equivalent to $\vee^3 S^1$, with $\partial M = \mathcal{S}_0(Wh(Bor))$. Its universal cover $\widetilde{M}$ is contractible. The end-point compactification of $\widetilde{M}$ is homeomorphic to the 4−ball. $\pi_1(M)$, the free group on three generators, acts on D^4.







Claim: There exist disjoint embeddings of six manifolds into D^4: three copies $\{A_i\}$ of A and three copies $\{B_i\}$ of B, such that $\alpha_1, \alpha_2, \alpha_3$ form the Borromean rings; $\beta_1, \beta_2, \beta_3$ are a parallel copy. This proves that the Borromean rings are weakly A-B slice.

Proof of the claim: a "relative-slice" problem. An illustration in 2 dimensions:

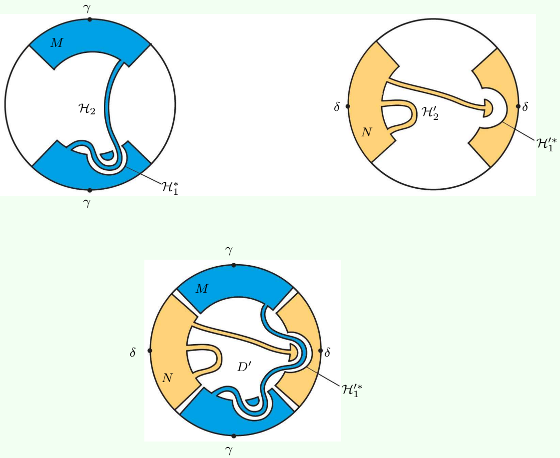
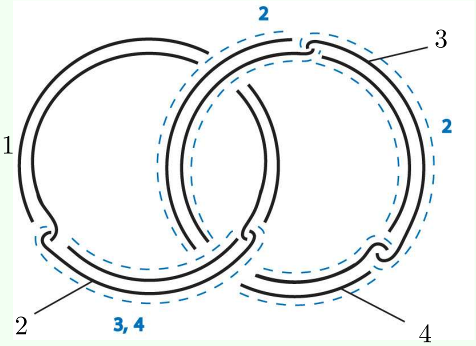


Figure: Disjoint embeddings of (M, γ), (N, δ) in (D^4, S^3), where γ, δ form a Hopf link in S^3.

There is a secondary obstruction, taking into account the embeddings $A \hookrightarrow D^4$, $B \hookrightarrow D^4$, showing that these decompositions do not solve the A-B slice problem.

The Hall-Witt identity:

$$[[x,y],z^x]\,[[z,x],y^z]\,[[y,z],x^y] = 1.$$



$$l_1 = [[m_3,m_4],m_2]\,[[m_2,m_3],m_4]\,[[m_4,m_2],m_3] = 1.$$

A link L is *homotopy* A-B slice if there exist decompositions A_i, B_i and disjoint maps $\alpha_i\colon\ A_i \longrightarrow D^4, \beta_i \longrightarrow D^4$ such that

(1) all sets in the collection $\alpha_1 A_1, \ldots, \alpha_n A_n, \beta_1 B_1, \ldots, \beta_n B_n$ are disjoint,

(2) the allowed singularities of the maps α_i, β_i are *self*-intersections of 2-handles of A_i, B_i (and not intersections between different 2-handles), and

(3) each map α_i, β_i is "link-homotopic" to the original embedding $A_i \hookrightarrow D^4, B_i \hookrightarrow D^4$.

Conjecture. *If the generalized Borromean rings are homotopy A-B slice then $Wh(Bor)$ is slice.*

Joint work with M. Freedman:
Consider $\mathcal{M} = \{(M,\gamma) | M$ is a codimension zero, smooth, compact submanifold of D^4, and $M \cap \partial D^4$ is a tubular neighborhood of an unknotted circle $\gamma \subset S^3\}$.

A **topological arbiter** is an invariant $\mathcal{A}\colon\ \mathcal{M} \longrightarrow \{0,1\}$ satisfying axioms (1) – (3):

(1) If (M,γ) is ambiently isotopic to (M',γ') in D^4 then $\mathcal{A}(M,\gamma) = \mathcal{A}(M',\gamma')$.

(2) If $(M,\gamma) \subset (M',\gamma)$ and $\mathcal{A}(M,\gamma) = 1$ then $\mathcal{A}(M',\gamma') = 1$.

(3) Let $D^4 = A \cup B$ be a decomposition of D^4, so the distinguished curves α, β of A, B form the Hopf link in ∂D^4. Then $\mathcal{A}(A,\alpha) + \mathcal{A}(B,\beta) = 1$.

Theorem There are uncountably many topological arbiters on D^4.

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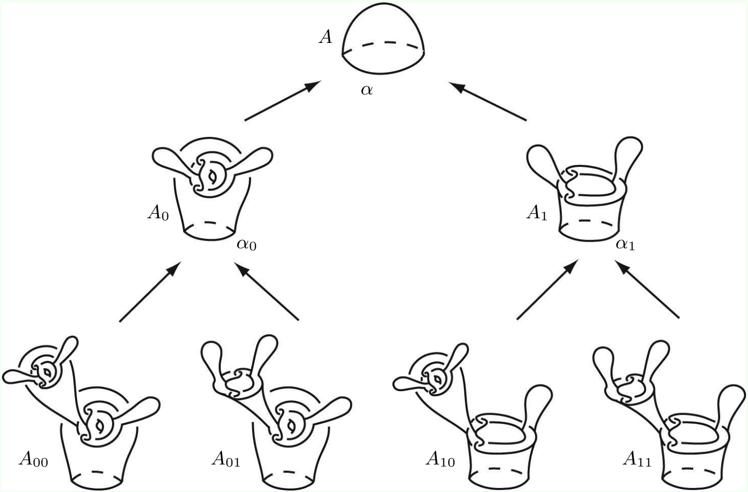
Axiom (4): Suppose $\mathcal{A}(M', \gamma') = 1$ and $\mathcal{A}(M'', \gamma';) = 1$. Then $\mathcal{A}(D(M', M''), \gamma) = 1$ where $D(M', M'')$ is the "Bing double" of M', M''.

Proposition A topological arbiter satisfying Axioms (1)-(4) is an obstruction to topological surgery.

Theorem There are uncountably many topological arbiters on D^4.



Theorem *Given a non-trivial square in the stable homotopy ring, there is a local topological arbiter not induced by homology on* D^{2n} *for sufficiently large* n.

For example, the Hopf map $h\colon S^3 \longrightarrow S^2$ is a generator of π_1^s, whose square is non-zero in π_2^s. There is an associated topological arbiter in dimension 8.