Parity criterion for unstabilized Heegaard splittings

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• handlebody

A handlebody can be obtained from a 3-ball by attaching 1-handles.
• Heegaard splitting

A Heegaard splitting $M = H_1 \cup_S H_2$ of a closed 3-manifold $M$ is a decomposition of $M$ into two handlebodies $H_1$ and $H_2$. ($S = \partial H_1 = \partial H_2$)

Every compact 3-manifold admits Heegaard splittings.
• stabilization

Add a trivial 1-handle to $H_1$ and remove it from $H_2$.

This results in a new Heegaard splitting $H'_1 \cup_{S'} H'_2$ with genus increased by one.
Equivalently,

A Heegaard splitting \( H_1 \cup_S H_2 \) obtained by a stabilization has essential disks \( D_1 \subset H_1 \) and \( D_2 \subset H_2 \) with \( |D_1 \cap D_2| = 1 \).
Example

[Waldhausen] Any positive genus Heegaard splitting of $S^3$ is stabilized.

We are interested in Heegaard splittings which are not stabilized. (unstabilized Heegaard splittings.)
For a genus $g \geq 2$ handlebody $H$, a collection of essential disks $\{D_1, \cdots, D_g\}$ in $H$ is called a **complete meridian disk system** if the result of cutting $H$ along $\bigcup_{i=1}^{g} D_i$ is a 3-ball.
We say that a collection of mutually disjoint essential disks \( \{D_1, \cdots, D_{3g-3}\} \) in \( H \) gives a **pants decomposition** if \( \bigcup_{i=1}^{3g-3} \partial D_i \) cuts \( \partial H \) into a collection of 2g – 2 pair of pants.
Theorem A

Let $M = H_1 \cup_S H_2$ be a genus $g \geq 2$ Heegaard splitting of a 3-manifold $M$ and $\{D_1, \ldots, D_g\}$ and $\{E_1, \ldots, E_g\}$ be complete meridian disk systems of $H_1$ and $H_2$, respectively.

If $|D_i \cap E_j| \equiv 0 \pmod{2}$ for all the pairs $(i, j)$, then $H_1 \cup_S H_2$ is unstabilized.
Lemma

Suppose that \( \{D_1, \ldots, D_g\} \) and \( \{E_1, \ldots, E_g\} \) satisfy that \( |D_i \cap E_j| \equiv 0 \pmod{2} \) for all \( 1 \leq i, j \leq g \).

Then there exist \textbf{pants decomposition} \( \{D_1, \ldots, D_g, D_{g+1}, \ldots, D_{3g-3}\} \) of \( H_1 \) and \( \{E_1, \ldots, E_g, E_{g+1}, \ldots, E_{3g-3}\} \) of \( H_2 \) such that \( |D_i \cap E_j| \equiv 0 \pmod{2} \) for all \( 1 \leq i, j \leq 3g - 3 \).
Theorem A’

Let $M = H_1 \cup_S H_2$ be a genus $g \geq 2$ Heegaard splitting of a 3-manifold $M$ and \{D_1, \cdots, D_{3g-3}\} and \{E_1, \cdots, E_{3g-3}\} give pants decomposition of $H_1$ and $H_2$, respectively.

If $|D_i \cap E_j| \equiv 0 \pmod{2}$ for all $1 \leq i, j \leq 3g - 3$, then $H_1 \cup_S H_2$ is unstabilized.
Sketch of proof)
Suppose it is stabilized. Then there exists disks $D$ in $H_1$ and $E$ in $H_2$ such that $|D \cap E| = 1$.

Cut $D$ by $\bigcup_{i=1}^{3g-3} D_i$ into subdisks and connect endpoints of arcs in $S$ as in the Figure. Do the same for $E$ with $\bigcup_{j=1}^{3g-3} E_j$. Then the parity of number of intersections of new curves is not changed.
Now $|\partial D \cap \partial E|$ is equivalent to $\sum |\partial D_i \cap \partial E_j|$ in (mod2) and by the hypothesis of Theorem A', it is 0 (mod2).

This contradicts that $|D \cap E| = 1$. 
• **Double (2-fold) branched covering**

Let \( L \) be an \( n \)-bridge link in \( S^3 \).

Let \( \{a_i\} \) be a collection of bridge arcs and \( \Delta_i \)'s be the corresponding disks in a 3-ball \( B_1 \) in an \( n \)-bridge presentation of \( L \).

Cut \( B_1 \) along \( \bigcup \Delta_i \) to get \( \bar{B}_1 \).

Cut \( B_1 \) along \( \bigcup \Delta_i \) to get \( \bar{B}_1 \).
Double $\tilde{B}^3_1$ along $\Delta_i$’s.

$\Rightarrow$ We obtain a genus $n - 1$ Heegaard splitting of the 2-fold branched cover of $S^3$ over $L$.
• **application**

• **Theorem**

Let $L$ be an $n$-component, $n$-bridge link in $S^3$. Then the induced Heegaard splitting of the 2-fold branched cover of $S^3$ over $L$ is unstabilized.

**Remark** There are examples (e.g. (among) torus knots) such that the Heegaard splitting for the 2-fold cover induced from minimal bridge presentation is stabilized.