An SU(3) Casson invariant for Rational Homology Spheres

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Historical Background

- 1990 Casson defined $\lambda : \{\mathbb{Z} \text{ homology spheres}\} \to \mathbb{Z}$ using $Hom(\pi_1 X, SU(2))$
- 1992 Taubes reinterpreted λ using $SU(2)$ gauge theory
- 1988 Floer defined instanton homology groups $IFH(X)$; Taubes showed $\chi(IFH(X^3)) = \lambda(X)$
- Walker extended λ to $\mathbb{Q}$HS's; Lescop extended it to all 3-manifolds
- 1998 Boden-H. gave $SU(3)$ generalization $\lambda_{SU(3)}$ for $\mathbb{Z}$HS's
  2001 Boden-H.-Kirk "renormalized" $\lambda_{SU(3)}$ to obtain simpler, integer-valued invariant
  2002 Cappell-Lee-Miller gave an alternative renormalization
- 2005 BHK calculated $\lambda_{SU(3)}$ for $\mathbb{Z}$HS surgeries on torus knots and other Brieskorn homology spheres

Motivation for current work

For $1/n$ surgery on (p,q)-torus knot,

$$\lambda_{SU(3)} = B(p,q)n + C(p,q)n^2,$$

where $C(p,q)$ is a certain Conway polynomial coefficient for the knot. $B(p,q)$ is not recognizable. More generally, calculations were done for Seifert fibered homology spheres $\Sigma(p,q,r)$.

$\lambda_{SU(3)}$ is not finite type, but perhaps it differs from a finite type by something we can recognize.

Extending $\lambda_{SU(3)}$ to rational homology spheres will give larger families of Seifert fibered manifold for which we can do similar calculations. This may help uncover conjectural pattern.

Outline

- Euler characteristic from a Morse function
- Strategy: analogous construction of invariants using Chern-Simons function
- Complications with analogous construction of Casson invariants from gauge theory
- Simplifications from homology restrictions
- Reductions for specific case of $SU(3)$ and $\mathbb{Q}$HS's
- Transversality and bifurcations in this case
- Definition of an invariant in this case

Morse theory analogy

Elementary fact:
$\chi(M^n) = \sum_{c \in crit(f)} (-1)^{\mu(c)}$ for any nondegenerate Morse function $f : M \to \mathbb{R}$.
A generic path from f_0 to f_1 gives a cobordism between $crit(f_0)$ and $crit(f_1)$.

$$W = \{(x,t) \mid x \in crit(f_t)\}$$

Gauge theoretic setup:
$\mathcal{A} = \{SU(n)$ connections on $E = X^3 \times \mathbb{C}^n\}$,
$cs : \mathcal{A} \to \mathbb{R}$, or $cs + h$, where h is a holonomy perturbation
Critical points of cs are flat connections

One potential problem (in two guises)

• Gauge group $\mathcal{G} = Aut(E)$ acts on $\mathcal{A}$ with varying orbit types; reducible connections $A = A_1 \oplus \cdots \oplus A_k$ on splitting $E = E_1 \oplus \cdots \oplus E_k$ have larger isotropy. $cs : \mathcal{A} \to \mathbb{R}$ is $\mathcal{G}$ invariant, so $crit(cs)$ consists of whole orbits. Worse than that, gauge invariance of $cs + h$ prevents transversality needed to get cobordism and invariance under perturbation.

• Consider $cs : \mathcal{A}/\mathcal{G} \to \mathbb{R}/\mathbb{Z}$. $crit(cs)$ equals $\mathcal{M} = \{\text{flat connections}\}/\mathcal{G} = Hom(\pi_1(X), SU(n))/\text{conjugation}$. But $\mathcal{B} = \mathcal{A}/\mathcal{G}$ has singularities at the orbits of reducible connections, and now singularities interfere with cobordism arguments.

Singularities interfere with cobordism arguments

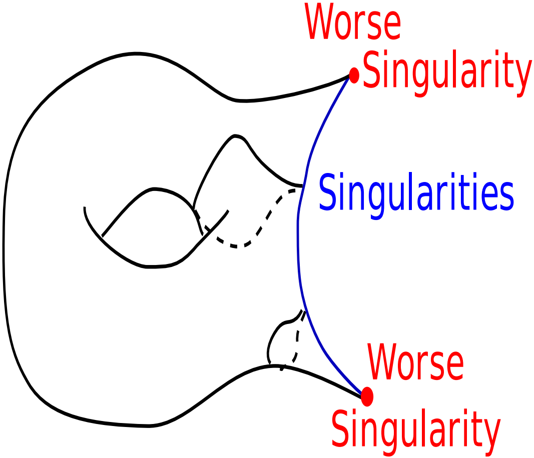

Figure: Varying function on a singular manifold does not produce a cobordism between critical sets. Critical points disappear into singular strata (and critical points in one stratum disappear into a worse stratum).

Homology restrictions

Homology restrictions on X limit abelian representations of $\pi_1(X)$; low rank $SU(n)$ limits the types of reductions.

$SU(2)$, $H_*(X;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$ implies $\mathcal{M} = \mathcal{M}^* \cup \{[\theta]\}$, where θ =trivial connection, and $\mathcal{M}^* = \{\text{irreducibles}\}$ is compact.

$$\sum_{\substack{[A] \in crit(cs+h) \\ [A] \neq [\theta]}} (-1)^{SF(\theta,A)} = \lambda_{SU(2)}(X)$$

Defined by Casson in terms of π_1 and Heegaard decomposition. Redefined as gauge theory Euler characteristic by Taubes, 1990, tying it to Floer's instanton homology.

Walker extended $\lambda_{SU(2)}$ to the case $H_*(X;\mathbb{Q}) = H_*(S^3;\mathbb{Q})$. Count irreducibles with sign, and add correction term from *abelian reps*, so the combination is perturbation invariant.

[Boden-H., B-H-Kirk]
For $SU(3)$ and $\mathbb{Z}$ homology spheres, $\mathcal{M} = \mathcal{M}^* \cup \mathcal{M}^{red} \cup \{[\theta]\}$, but $\{[\theta]\}$ is isolated. A correction term involving reducibles of the form $A = A_1 \oplus A_2$ compensates for the dependence of
$$\sum_{[A]\in crit(cs+h)\text{ irred}} (-1)^{SF(\theta,A)}$$ on the perturbation.

Correction term details for $2 \oplus 1$ connections

SF denotes spectral flow of the twisted signature operator, which amounts to a relative Morse index between two critical points in this ∞-dimensional context.

$$\lambda_{SU(3)}(X) = \sum_{[A]\in\mathcal{M}_h^*} (-1)^{SF(\theta,A)} - \sum_{[B]\in\mathcal{M}_h^{red}} (-1)^{SF(\theta,B)} \left[\frac{SF_N(B_0,B)}{2}\right].$$

Here, B_0 is a suitably chosen reducible basepoint (depending on component of moduli space M^{red} B arises from).

Decompose spectral flow (along a path of reducible connections) into "tangent to reducibles" and "normal to reducibles" components. SF_N is the latter.

Extending $\lambda_{SU(3)}$ to $\mathbb{Q}$HS case

(joint work in progress with Hans Boden)
Consider $SU(3)$ connections on a rational homology sphere X.

Orbit types (i.e., singular strata in $\mathcal{A}/\mathcal{G}$):
irreducible,
$2 \oplus 1$ (i.e., $A_1 \oplus A_2$, of ranks 2 and 1, resp.),
$1 \oplus 1 \oplus 1$ (i.e., sum of distinct rank 1 connections),
$1 \oplus 1^2$ (i.e., $A_1 \oplus A_2 \oplus A_2$), and
1^3=central.

Bifurcations and the parameterized moduli space

What effect do these different types of singularities play? How can the critical set change (besides by a cobordism, preserving the number of points counted with sign)?

We keep track of changes in topology of the (perturbed) flat moduli space by working with the *parameterized moduli space*, $W = \{([A], t) \in \mathcal{B} \mid grad(cs + h_t)(A) = 0\}$, for any path $h_t, 0 \leq t \leq 1$.

Structure of the parameterized moduli space

The structure of $W = \{([A], t) \in \mathcal{B} \mid grad(cs + h_t)(A) = 0\}$ for a generic path $h_t, 0 \leq t \leq 1$ is described in *Transversality for equivariant exact 1-forms and gauge theory on 3 manifolds*, H., AIM 2006.

$\mathbb{Q}$HS restriction implies abelians are isolated from one another. This shows $W^{1\oplus 1\oplus 1}$, $W^{1^2\oplus 1}$ and W^{1^3} form compact product cobordisms.

$W^{2\oplus 1}$ is a compact cobordism except for ends hitting $W^{1\oplus 1\oplus 1}$ and $W^{1^2\oplus 1}$.

W^* is compact except for ends hitting $W^{2\oplus 1}$.

The cobordism will have 3 types of singularities, all modeled on T-intersections.

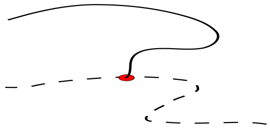

Figure: A T-intersection.

Levels of reducibility

Solid	Dashed	Description
Irreducible	$2 \oplus 1$	W^* runs into $W^{2\oplus 1}$
$2 \oplus 1$	$1 \oplus 1 \oplus 1$	$W^{2\oplus 1}$ runs into $W^{1\oplus 1\oplus 1}$
$2 \oplus 1$	$1^2 \oplus 1$	$W^{2\oplus 1}$ runs into $W^{1^2\oplus 1}$

Correction terms to account for bifurcations

Irreducible critical points can pop out of $2 \oplus 1$ critical points. In addition, $2 \oplus 1$'s can pop out of the $1 \oplus 1 \oplus 1$ or $1 \oplus 1^2$ strata.

$\sum_{\mathcal{M}_h^*} (-1)^{SF(\theta,A)}$ is changes when irreducibles can pop out of reducibles, as perturbation is varied. We need [BHK] correction term for $2 \oplus 1$ stratum.

The [BHK] correction term also changes (in a different way than adding ± 1) when a $2 \oplus 1$ point pops out of the lower stratum, so we need $1 \oplus 1 \oplus 1$ and $1 \oplus 1^2$ correction terms that account for this.

Normal spectral flow along abelians

Consider a path of abelian con's $C(t) = C_1(t) \oplus C_2(t) \oplus C_3(t)$ on $E = E_1 \oplus E_2 \oplus E_3$.
$T_{C(t)}\{\text{abelian connections}\} = \Omega^1(X; diag(su(3))$.
Normal bundle is $\Omega^1(X; \mathbb{C}^3)$.

$$\mathbb{C}^3 = \left\{ \begin{bmatrix} 0 & z_{12} & z_{13} \\ -\overline{z}_{12} & 0 & z_{23} \\ -\overline{z}_{13} & -\overline{z}_{23} & 0 \end{bmatrix} \middle| (z_{12}, z_{13}, z_{23}) \in \mathbb{C}^3 \right\}$$

Abelian correction term

The $1 \oplus 1 \oplus 1$ correction term is essentially

$$\frac{1}{4} \sum_{[C] \in \mathcal{M}^{1\oplus 1 \oplus 1}} (-1)^{SF(\theta,C)} \left[SF_{12}(C_0, C) SF_{13}(C_0, C) \right.$$
$$\left. SF_{12}(C_0, C) SF_{23}(C_0, C) + SF_{13}(C_0, C) SF_{23}(C_0, C) \right].$$

More precisely, for each $2 \oplus 1$ splitting of $E = M_{-\omega} \oplus L_\omega$, each $C = (C_1 \oplus C_2) \oplus C_3)$ contributes

$$\tfrac{1}{4}(-1)^{SF(\theta,C)} \left[SF_{12}(C_0, C) SF_N(B_0, C) \right]$$

where SF_N means normal to $M_{-\omega} \oplus L_\omega$ reducibles.

There is an analogous correction term for $1 \oplus 1^2$ points.

Formula

Theorem
The following quantity is independent of perturbation, and so is an invariant of rational homology spheres X. It reduces to the [BHK] invariant for $\mathbb{Z}$HS's or for $\mathbb{Q}$HS's where the abelians are all non-degenerate.

$$\lambda_{SU(3)}(X) = \sum_{[A]\in\mathcal{M}_h^*} (-1)^{SF(\theta,A)} - \sum_{[B]\in\mathcal{M}_h^{red}} (-1)^{SF(\theta,B)} \left[\frac{SF_N(B_0,B)}{2}\right]$$

$$+\tfrac{1}{4} \sum_{[C]\in\mathcal{M}_h^{1\oplus1\oplus1}} (-1)^{SF(\theta,C)} \left[SF_{12}(C_0,C)SF_N(B_0,C)\right]$$

$$+\tfrac{1}{4} \sum_{[C]\in\mathcal{M}_h^{1\oplus1^2}} (-1)^{SF(\theta,C)} \left[\ldots\right]$$