Twisted Alexander polynomials of hyperbolic knots

Stefan Friedl
joint with N. Dunfield, N. Jackson and S. Vidussi

Seoul 2009

The classical Alexander polynomial

Let $K \subset S^3$. The symmetrized classical Alexander polynomial $\Delta_K(t)$ has the following formal properties:

The classical Alexander polynomial

Let $K \subset S^3$. The symmetrized classical Alexander polynomial $\Delta_K(t)$ has the following formal properties:
(1) $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$,

The classical Alexander polynomial

Let $K \subset S^3$. The symmetrized classical Alexander polynomial $\Delta_K(t)$ has the following formal properties:
(1) $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$,
(2) $\Delta_K(1) = 1$,

The classical Alexander polynomial

Let $K \subset S^3$. The symmetrized classical Alexander polynomial $\Delta_K(t)$ has the following formal properties:
(1) $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$,
(2) $\Delta_K(1) = 1$,
(3) $\Delta_K(t^{-1}) = \Delta_K(t)$.

The classical Alexander polynomial

Let $K \subset S^3$. The symmetrized classical Alexander polynomial $\Delta_K(t)$ has the following formal properties:
(1) $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$,
(2) $\Delta_K(1) = 1$,
(3) $\Delta_K(t^{-1}) = \Delta_K(t)$.
In fact any polynomial satisfying (1), (2) and (3) appears as the Alexander polynomial of a knot K.

The classical Alexander polynomial

Let $K \subset S^3$. The symmetrized classical Alexander polynomial $\Delta_K(t)$ has the following formal properties:
(1) $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$,
(2) $\Delta_K(1) = 1$,
(3) $\Delta_K(t^{-1}) = \Delta_K(t)$.

Note that (1), (2) and (3) also imply
(2') $\Delta_K(\xi) \neq 0$ for any prime power root of unity ξ.

The classical Alexander polynomial

Let $K \subset S^3$. The symmetrized classical Alexander polynomial $\Delta_K(t)$ has the following formal properties:
(1) $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$,
(2) $\Delta_K(1) = 1$,
(3) $\Delta_K(t^{-1}) = \Delta_K(t)$.

Note that (1), (2) and (3) also imply
(2') $\Delta_K(\xi) \neq 0$ for any prime power root of unity ξ.
The Alexander polynomial $\Delta_K(t)$ also contains topological information:

The classical Alexander polynomial

Let $K \subset S^3$. The symmetrized classical Alexander polynomial $\Delta_K(t)$ has the following formal properties:
(1) $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$,
(2) $\Delta_K(1) = 1$,
(3) $\Delta_K(t^{-1}) = \Delta_K(t)$.

Note that (1), (2) and (3) also imply
(2') $\Delta_K(\xi) \neq 0$ for any prime power root of unity ξ.
The Alexander polynomial $\Delta_K(t)$ also contains topological information:
(4) $\Delta_K(t)$ does not depend on the orientation,

The classical Alexander polynomial

Let $K \subset S^3$. The symmetrized classical Alexander polynomial $\Delta_K(t)$ has the following formal properties:
(1) $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$,
(2) $\Delta_K(1) = 1$,
(3) $\Delta_K(t^{-1}) = \Delta_K(t)$.

Note that (1), (2) and (3) also imply
(2') $\Delta_K(\xi) \neq 0$ for any prime power root of unity ξ.
The Alexander polynomial $\Delta_K(t)$ also contains topological information:
(4) $\Delta_K(t)$ does not depend on the orientation,
(5) $\Delta_{K^*}(t) = \Delta_K(t)$ where K^* is the mirror image,

The classical Alexander polynomial

Let $K \subset S^3$. The symmetrized classical Alexander polynomial $\Delta_K(t)$ has the following formal properties:
(1) $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$,
(2) $\Delta_K(1) = 1$,
(3) $\Delta_K(t^{-1}) = \Delta_K(t)$.

Note that (1), (2) and (3) also imply
(2') $\Delta_K(\xi) \neq 0$ for any prime power root of unity ξ.
The Alexander polynomial $\Delta_K(t)$ also contains topological information:
(4) $\Delta_K(t)$ does not depend on the orientation,
(5) $\Delta_{K^*}(t) = \Delta_K(t)$ where K^* is the mirror image,
(6) $\Delta_K(t)$ is invariant under mutation,

The classical Alexander polynomial

Let $K \subset S^3$. The symmetrized classical Alexander polynomial $\Delta_K(t)$ has the following formal properties:
(1) $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$,
(2) $\Delta_K(1) = 1$,
(3) $\Delta_K(t^{-1}) = \Delta_K(t)$.

Note that (1), (2) and (3) also imply
(2') $\Delta_K(\xi) \neq 0$ for any prime power root of unity ξ.
The Alexander polynomial $\Delta_K(t)$ also contains topological information:
(4) $\Delta_K(t)$ does not depend on the orientation,
(5) $\Delta_{K^*}(t) = \Delta_K(t)$ where K^* is the mirror image,
(6) $\Delta_K(t)$ is invariant under mutation,
(7) $\deg(\Delta_K(t)) \leq 2$genus of K,

The classical Alexander polynomial

Let $K \subset S^3$. The symmetrized classical Alexander polynomial $\Delta_K(t)$ has the following formal properties:
(1) $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$,
(2) $\Delta_K(1) = 1$,
(3) $\Delta_K(t^{-1}) = \Delta_K(t)$.

Note that (1), (2) and (3) also imply
(2') $\Delta_K(\xi) \neq 0$ for any prime power root of unity ξ.
The Alexander polynomial $\Delta_K(t)$ also contains topological information:
(4) $\Delta_K(t)$ does not depend on the orientation,
(5) $\Delta_{K^*}(t) = \Delta_K(t)$ where K^* is the mirror image,
(6) $\Delta_K(t)$ is invariant under mutation,
(7) $\deg(\Delta_K(t)) \leq 2$genus of K,
(8) if K is fibered, then $\Delta_K(t)$ is monic.

The classical Alexander polynomial and Fox calculus

Let

$$\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_{k-1} \rangle$$

be a presentation of $\pi_1(S^3 \setminus K)$. Denote by $\phi : \pi \to \langle t \rangle$ the epimorphism.

The classical Alexander polynomial and Fox calculus

Let

$$\pi = \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_{k-1} \rangle$$

be a presentation of $\pi_1(S^3 \setminus K)$. Denote by $\phi : \pi \to \langle t \rangle$ the epimorphism.Pick i with $\phi(g_i) \neq 0$, then

$$\frac{\Delta_K(t)}{t-1} = \frac{\det(\phi(\text{matrix } \left(\frac{\partial r_k}{\partial g_l}\right) \text{with } i\text{-th column removed}))}{\phi(g_i) - 1}.$$

Let

$$\pi = \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_{k-1} \rangle$$

be a presentation of $\pi_1(S^3 \setminus K)$. Denote by $\phi : \pi \to \langle t \rangle$ the epimorphism.Pick i with $\phi(g_i) \neq 0$, then

$$\frac{\Delta_K(t)}{t-1} = \frac{\det(\phi(\text{matrix } \left(\frac{\partial r_k}{\partial g_l}\right) \text{with } i\text{-th column removed}))}{\phi(g_i) - 1}.$$

Note that the formula on the right hand side really computes the Reidemeister torsion of $C_*(S^3 \setminus K, \mathbb{Q}(t))$.

The twisted Alexander polynomial

Let $K \subset S^3$ be a knot and $\alpha : \pi \to \mathrm{SL}(n, \mathbb{C})$ a representation.

The twisted Alexander polynomial

Let $K \subset S^3$ be a knot and $\alpha : \pi \to \mathrm{SL}(n, \mathbb{C})$ a representation.Denote by $\phi : \pi \to \langle t \rangle$ the epimorphism.

The twisted Alexander polynomial

Let $K \subset S^3$ be a knot and $\alpha : \pi \to \mathrm{SL}(n, \mathbb{C})$ a representation.Denote by $\phi : \pi \to \langle t \rangle$ the epimorphism. Let

$$\pi = \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_{k-1} \rangle$$

be a presentation.

The twisted Alexander polynomial

Let $K \subset S^3$ be a knot and $\alpha : \pi \to \mathrm{SL}(n, \mathbb{C})$ a representation.Denote by $\phi : \pi \to \langle t \rangle$ the epimorphism. Let

$$\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_{k-1} \rangle$$

be a presentation.Pick i with $\phi(g_i) \neq 0$, then we *define*

$$\tau(K, \alpha) := \frac{\det((\alpha \otimes \phi)(\text{matrix } \left(\frac{\partial r_k}{\partial g_l}\right) \text{with } i\text{-th column removed}))}{(\alpha \otimes \phi)(g_i) - 1}.$$

The twisted Alexander polynomial

Let $K \subset S^3$ be a knot and $\alpha : \pi \to \mathrm{SL}(n, \mathbb{C})$ a representation.Denote by $\phi : \pi \to \langle t \rangle$ the epimorphism. Let

$$\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_{k-1} \rangle$$

be a presentation.Pick i with $\phi(g_i) \neq 0$, then we *define*

$$\tau(K, \alpha) := \frac{\det((\alpha \otimes \phi)(\text{matrix } \left(\frac{\partial r_k}{\partial g_l}\right) \text{with } i\text{-th column removed}))}{(\alpha \otimes \phi)(g_i) - 1}.$$

This invariant is sometimes referred to as:

The twisted Alexander polynomial

Let $K \subset S^3$ be a knot and $\alpha : \pi \to \mathrm{SL}(n, \mathbb{C})$ a representation.Denote by $\phi : \pi \to \langle t \rangle$ the epimorphism. Let

$$\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_{k-1} \rangle$$

be a presentation.Pick i with $\phi(g_i) \neq 0$, then we *define*

$$\tau(K, \alpha) := \frac{\det((\alpha \otimes \phi)(\text{matrix } \left(\frac{\partial r_k}{\partial g_l}\right) \text{ with } i\text{-th column removed}))}{(\alpha \otimes \phi)(g_i) - 1}.$$

This invariant is sometimes referred to as:
(1) twisted Alexander polynomial,

The twisted Alexander polynomial

Let $K \subset S^3$ be a knot and $\alpha : \pi \to \mathrm{SL}(n, \mathbb{C})$ a representation.Denote by $\phi : \pi \to \langle t \rangle$ the epimorphism. Let

$$\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_{k-1} \rangle$$

be a presentation.Pick i with $\phi(g_i) \neq 0$, then we *define*

$$\tau(K, \alpha) := \frac{\det((\alpha \otimes \phi)(\text{matrix } \left(\frac{\partial r_k}{\partial g_l}\right) \text{with } i\text{-th column removed}))}{(\alpha \otimes \phi)(g_i) - 1}.$$

This invariant is sometimes referred to as:
(1) twisted Alexander polynomial,
(2) Wada's invariant,

The twisted Alexander polynomial

Let $K \subset S^3$ be a knot and $\alpha : \pi \to \mathrm{SL}(n, \mathbb{C})$ a representation.Denote by $\phi : \pi \to \langle t \rangle$ the epimorphism. Let

$$\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_{k-1} \rangle$$

be a presentation.Pick i with $\phi(g_i) \neq 0$, then we *define*

$$\tau(K, \alpha) := \frac{\det((\alpha \otimes \phi)(\text{matrix } \left(\frac{\partial r_k}{\partial g_l}\right) \text{ with } i\text{-th column removed}))}{(\alpha \otimes \phi)(g_i) - 1}.$$

This invariant is sometimes referred to as:
(1) twisted Alexander polynomial,
(2) Wada's invariant,
(3) twisted Reidemeister torsion.

The twisted Alexander polynomial

Let $K \subset S^3$ be a knot and $\alpha : \pi \to \mathrm{SL}(n, \mathbb{C})$ a representation.Denote by $\phi : \pi \to \langle t \rangle$ the epimorphism. Let

$$\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_{k-1} \rangle$$

be a presentation.Pick i with $\phi(g_i) \neq 0$, then we *define*

$$\tau(K, \alpha) := \frac{\det((\alpha \otimes \phi)(\text{matrix } \left(\frac{\partial r_k}{\partial g_l}\right) \text{ with } i\text{-th column removed}))}{(\alpha \otimes \phi)(g_i) - 1}.$$

This invariant is sometimes referred to as:
(1) twisted Alexander polynomial,
(2) Wada's invariant,
(3) twisted Reidemeister torsion.

Theorem. (Wada, Goda-Kitano-Morifuji) If α is an even dimensional representation, then $\tau(K, \alpha)$ is well-defined up to multiplication by $t^i, i \in \mathbb{Z}$.

Hyperbolic knots

Let $K \subset S^3$ be an oriented hyperbolic knot. We denote by μ, λ its meridian and longitude. There exists a discrete and faithful repr.

$$\alpha : \pi_1(S^3 \setminus K) \to \mathrm{SL}(2, \mathbb{C})$$

Hyperbolic knots

Let $K \subset S^3$ be an oriented hyperbolic knot. We denote by μ, λ its meridian and longitude. There exists a discrete and faithful repr.

$$\alpha : \pi_1(S^3 \setminus K) \to \mathrm{SL}(2, \mathbb{C})$$

which is unique up to conjugation if we demand that

$$\alpha(\mu) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \alpha(\lambda) = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$$

for some a with $\mathrm{Im}(a) > 0$.

Hyperbolic knots

Let $K \subset S^3$ be an oriented hyperbolic knot. We denote by μ, λ its meridian and longitude. There exists a discrete and faithful repr.

$$\alpha : \pi_1(S^3 \setminus K) \to \mathrm{SL}(2, \mathbb{C})$$

which is unique up to conjugation if we demand that

$$\alpha(\mu) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \alpha(\lambda) = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$$

for some a with $\mathrm{Im}(a) > 0$.

Theorem. (1) $\tau(K, \alpha)$ is well-defined up to multiplication by $t^i, i \in \mathbb{Z}$. (Wada, Goda-Kitano-Morifuji)

Hyperbolic knots

Let $K \subset S^3$ be an oriented hyperbolic knot. We denote by μ, λ its meridian and longitude. There exists a discrete and faithful repr.

$$\alpha : \pi_1(S^3 \setminus K) \to \mathrm{SL}(2, \mathbb{C})$$

which is unique up to conjugation if we demand that

$$\alpha(\mu) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \alpha(\lambda) = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$$

for some a with $\mathrm{Im}(a) > 0$.

Theorem. (1) $\tau(K, \alpha)$ is well-defined up to multiplication by $t^i, i \in \mathbb{Z}$. (Wada, Goda-Kitano-Morifuji)

(2) $\tau(K, \alpha)$ is palindromic, i.e. $\tau(K, \alpha)(t^{-1}) = \pm t^k \tau(K, \alpha)$ (Hillman-Silver-Williams)

Hyperbolic knots

Let $K \subset S^3$ be an oriented hyperbolic knot. We denote by μ, λ its meridian and longitude. There exists a discrete and faithful repr.

$$\alpha : \pi_1(S^3 \setminus K) \to \mathrm{SL}(2, \mathbb{C})$$

which is unique up to conjugation if we demand that

$$\alpha(\mu) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \alpha(\lambda) = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$$

for some a with $\mathrm{Im}(a) > 0$.

Theorem. (1) $\tau(K, \alpha)$ is well-defined up to multiplication by $t^i, i \in \mathbb{Z}$. (Wada, Goda-Kitano-Morifuji)
(2) $\tau(K, \alpha)$ is palindromic, i.e. $\tau(K, \alpha)(t^{-1}) = \pm t^k \tau(K, \alpha)$ (Hillman-Silver-Williams)
(3) $\tau(K, \alpha)(t = 1)$ is non-zero (Menal-Ferrer and Porti)

Hyperbolic knots

Let $K \subset S^3$ be an oriented hyperbolic knot. We denote by μ, λ its meridian and longitude. There exists a discrete and faithful repr.

$$\alpha : \pi_1(S^3 \setminus K) \to \mathrm{SL}(2, \mathbb{C})$$

which is unique up to conjugation if we demand that

$$\alpha(\mu) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \alpha(\lambda) = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$$

for some a with $\mathrm{Im}(a) > 0$.

Theorem. (1) $\tau(K, \alpha)$ is well-defined up to multiplication by $t^i, i \in \mathbb{Z}$. (Wada, Goda-Kitano-Morifuji)
(2) $\tau(K, \alpha)$ is palindromic, i.e. $\tau(K, \alpha)(t^{-1}) = \pm t^k \tau(K, \alpha)$ (Hillman-Silver-Williams)
(3) $\tau(K, \alpha)(t = 1)$ is non-zero (Menal-Ferrer and Porti)
(4) $\tau(K, \alpha)$ is a polynomial in $\mathbb{C}[t^{\pm 1}]$ of even degree.

The invariant $\mathcal{T}_K(t)$

In light of the previous theorem we can now define

$$\mathcal{T}_K(t) := \text{symmetrization of } \tau(K,\alpha).$$

The invariant $\mathcal{T}_K(t)$

In light of the previous theorem we can now define

$$\mathcal{T}_K(t) := \text{symmetrization of } \tau(K,\alpha).$$

Theorem. (1) $\mathcal{T}_K(t)$ is a well-defined polynomial in $\mathbb{C}[t^{\pm 1}]$,

The invariant $\mathcal{T}_K(t)$

In light of the previous theorem we can now define

$$\mathcal{T}_K(t) := \text{symmetrization of } \tau(K, \alpha).$$

Theorem. (1) $\mathcal{T}_K(t)$ is a well-defined polynomial in $\mathbb{C}[t^{\pm 1}]$,
(2) $\mathcal{T}_K(1) \neq 0$, in fact $\mathcal{T}_K(\xi) \neq 0$ for any root of unity ξ,

The invariant $\mathcal{T}_K(t)$

In light of the previous theorem we can now define

$$\mathcal{T}_K(t) := \text{symmetrization of } \tau(K, \alpha).$$

Theorem. (1) $\mathcal{T}_K(t)$ is a well-defined polynomial in $\mathbb{C}[t^{\pm 1}]$,
(2) $\mathcal{T}_K(1) \neq 0$, in fact $\mathcal{T}_K(\xi) \neq 0$ for any root of unity ξ,
(follows from results of Menal-Ferrer and Porti)

The invariant $\mathcal{T}_K(t)$

In light of the previous theorem we can now define

$$\mathcal{T}_K(t) := \text{symmetrization of } \tau(K, \alpha).$$

Theorem. (1) $\mathcal{T}_K(t)$ is a well-defined polynomial in $\mathbb{C}[t^{\pm 1}]$,
(2) $\mathcal{T}_K(1) \neq 0$, in fact $\mathcal{T}_K(\xi) \neq 0$ for any root of unity ξ,
(3) $\mathcal{T}_K(t^{-1}) = \mathcal{T}_K(t)$,

The invariant $\mathcal{T}_K(t)$

In light of the previous theorem we can now define

$$\mathcal{T}_K(t) := \text{symmetrization of } \tau(K, \alpha).$$

Theorem. (1) $\mathcal{T}_K(t)$ is a well-defined polynomial in $\mathbb{C}[t^{\pm 1}]$,
(2) $\mathcal{T}_K(1) \neq 0$, in fact $\mathcal{T}_K(\xi) \neq 0$ for any root of unity ξ,
(3) $\mathcal{T}_K(t^{-1}) = \mathcal{T}_K(t)$,
(4) $\mathcal{T}_K(t)$ does *not* depend on the orientation of K,

The invariant $\mathcal{T}_K(t)$

In light of the previous theorem we can now define

$$\mathcal{T}_K(t) := \text{symmetrization of } \tau(K,\alpha).$$

Theorem. (1) $\mathcal{T}_K(t)$ is a well-defined polynomial in $\mathbb{C}[t^{\pm 1}]$,
(2) $\mathcal{T}_K(1) \neq 0$, in fact $\mathcal{T}_K(\xi) \neq 0$ for any root of unity ξ,
(3) $\mathcal{T}_K(t^{-1}) = \mathcal{T}_K(t)$,
(4) $\mathcal{T}_K(t)$ does *not* depend on the orientation of K,
(5) $\mathcal{T}_{K^*}(t) = \overline{\mathcal{T}_K(t)}$,

The invariant $\mathcal{T}_K(t)$

In light of the previous theorem we can now define

$$\mathcal{T}_K(t) := \text{symmetrization of } \tau(K, \alpha).$$

Theorem. (1) $\mathcal{T}_K(t)$ is a well-defined polynomial in $\mathbb{C}[t^{\pm 1}]$,
(2) $\mathcal{T}_K(1) \neq 0$, in fact $\mathcal{T}_K(\xi) \neq 0$ for any root of unity ξ,
(3) $\mathcal{T}_K(t^{-1}) = \mathcal{T}_K(t)$,
(4) $\mathcal{T}_K(t)$ does *not* depend on the orientation of K,
(5) $\mathcal{T}_{K^*}(t) = \overline{\mathcal{T}_K(t)}$, in particular if K is amphichiral, then $\mathcal{T}_K(t)$ is a real polynomial.

The invariant $\mathcal{T}_K(t)$

In light of the previous theorem we can now define

$$\mathcal{T}_K(t) := \text{symmetrization of } \tau(K,\alpha).$$

Theorem. (1) $\mathcal{T}_K(t)$ is a well-defined polynomial in $\mathbb{C}[t^{\pm 1}]$,
(2) $\mathcal{T}_K(1) \neq 0$, in fact $\mathcal{T}_K(\xi) \neq 0$ for any root of unity ξ,
(3) $\mathcal{T}_K(t^{-1}) = \mathcal{T}_K(t)$,
(4) $\mathcal{T}_K(t)$ does *not* depend on the orientation of K,
(5) $\mathcal{T}_{K^*}(t) = \overline{\mathcal{T}_K(t)}$, in particular if K is amphichiral, then $\mathcal{T}_K(t)$ is a real polynomial.

Example.

$$\mathcal{T}_{\text{Figure 8 knot}}(t) = t - 4 + t^{-1}.$$

The invariant $\mathcal{T}_K(t)$

In light of the previous theorem we can now define

$$\mathcal{T}_K(t) := \text{symmetrization of } \tau(K, \alpha).$$

Theorem. (1) $\mathcal{T}_K(t)$ is a well-defined polynomial in $\mathbb{C}[t^{\pm 1}]$,
(2) $\mathcal{T}_K(1) \neq 0$, in fact $\mathcal{T}_K(\xi) \neq 0$ for any root of unity ξ,
(3) $\mathcal{T}_K(t^{-1}) = \mathcal{T}_K(t)$,
(4) $\mathcal{T}_K(t)$ does *not* depend on the orientation of K,
(5) $\mathcal{T}_{K^*}(t) = \overline{\mathcal{T}_K(t)}$, in particular if K is amphichiral, then $\mathcal{T}_K(t)$ is a real polynomial.

Example.

$$\mathcal{T}_{\text{Figure 8 knot}}(t) = t - 4 + t^{-1}.$$

Remark. If K is a hyperbolic knot with at most thirteen crossings, then K is amphichiral if and only if $\mathcal{T}_K(t)$ is a real polynomial.

The invariant $\mathcal{T}_K(t)$

In light of the previous theorem we can now define

$$\mathcal{T}_K(t) := \text{symmetrization of } \tau(K,\alpha).$$

Theorem. (1) $\mathcal{T}_K(t)$ is a well-defined polynomial in $\mathbb{C}[t^{\pm 1}]$,
(2) $\mathcal{T}_K(1) \neq 0$, in fact $\mathcal{T}_K(\xi) \neq 0$ for any root of unity ξ,
(3) $\mathcal{T}_K(t^{-1}) = \mathcal{T}_K(t)$,
(4) $\mathcal{T}_K(t)$ does *not* depend on the orientation of K,
(5) $\mathcal{T}_{K^*}(t) = \overline{\mathcal{T}_K(t)}$, in particular if K is amphichiral, then $\mathcal{T}_K(t)$ is a real polynomial.

Example.

$$\mathcal{T}_{\text{Figure 8 knot}}(t) = t - 4 + t^{-1}.$$

Remark. If K is a hyperbolic knot with at most thirteen crossings, then K is amphichiral if and only if $\mathcal{T}_K(t)$ is a real polynomial.

Remark. If K is an amphichiral hyperbolic knot with at most thirteen crossings, then the top coefficient of $\mathcal{T}_K(t)$ is at least one.

The Kinoshita-Terasaka knot and the Conway knot

The Conway knot and the Kinoshita-Terasaka knot are mutants knots with trivial Alexander polynomial.

The Kinoshita-Terasaka knot and the Conway knot

The Conway knot and the Kinoshita-Terasaka knot are mutants knots with trivial Alexander polynomial.We calculate

$$\begin{aligned}\mathcal{T}_{Conway}(t) \approx \quad & (4.895 - 0.099i)t^5 \qquad +(-15.686 + 0.297i)t^4 \\ & + \dots (-15.686 + 0.298i)t^{-4} + (4.895 - 0.099i)t^{-5}.\end{aligned}$$

The Kinoshita-Terasaka knot and the Conway knot

The Conway knot and the Kinoshita-Terasaka knot are mutants knots with trivial Alexander polynomial.We calculate

$$\begin{aligned}\mathcal{T}_{Conway}(t) \approx \quad & (4.895-0.099i)t^5 \qquad +(-15.686+0.297i)t^4 \\ & +\dots(-15.686+0.298i)t^{-4} +(4.895-0.099i)t^{-5}.\end{aligned}$$

For the Kinoshita–Terasaka knot we calculate

$$\begin{aligned}\mathcal{T}_{KT}(t) \approx \quad & (4.418+0.376i)t^3 \qquad +(-22.942-4.845i)t^2 \\ & +\dots+(-22.926-4.845i)t^{-2} \quad +(4.418+0.376i)t^{-3}.\end{aligned}$$

The Kinoshita-Terasaka knot and the Conway knot

The Conway knot and the Kinoshita-Terasaka knot are mutants knots with trivial Alexander polynomial.We calculate

$$\begin{aligned} \mathcal{T}_{Conway}(t) \approx \; & (4.895 - 0.099i)t^5 & +(-15.686 + 0.297i)t^4 \\ & + \dots (-15.686 + 0.298i)t^{-4} & +(4.895 - 0.099i)t^{-5}. \end{aligned}$$

For the Kinoshita–Terasaka knot we calculate

$$\begin{aligned} \mathcal{T}_{KT}(t) \approx \; & (4.418 + 0.376i)t^3 & +(-22.942 - 4.845i)t^2 \\ & + \cdots + (-22.926 - 4.845i)t^{-2} & +(4.418 + 0.376i)t^{-3}. \end{aligned}$$

This shows that $\mathcal{T}_K(t)$ detects mutation.

The Kinoshita-Terasaka knot and the Conway knot

The Conway knot and the Kinoshita-Terasaka knot are mutants knots with trivial Alexander polynomial.We calculate

$$\begin{aligned} \mathcal{T}_{Conway}(t) \approx \; & (4.895 - 0.099i)t^5 & +(-15.686 + 0.297i)t^4 \\ & + \dots (-15.686 + 0.298i)t^{-4} & +(4.895 - 0.099i)t^{-5}. \end{aligned}$$

For the Kinoshita–Terasaka knot we calculate

$$\begin{aligned} \mathcal{T}_{KT}(t) \approx \; & (4.418 + 0.376i)t^3 & +(-22.942 - 4.845i)t^2 \\ & + \cdots + (-22.926 - 4.845i)t^{-2} & +(4.418 + 0.376i)t^{-3}. \end{aligned}$$

This shows that $\mathcal{T}_K(t)$ detects mutation.The evaluations $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ conjecturally do *not* detect mutation. E.g. for both knots evaluation at $t = 1$ gives $4.1860 - 4.2286i$

The Kinoshita-Terasaka knot and the Conway knot

The Conway knot and the Kinoshita-Terasaka knot are mutants knots with trivial Alexander polynomial.We calculate

$$\mathcal{T}_{Conway}(t) \approx \begin{array}{ll} (4.895 - 0.099i)t^5 & +(-15.686 + 0.297i)t^4 \\ + \dots (-15.686 + 0.298i)t^{-4} & +(4.895 - 0.099i)t^{-5}. \end{array}$$

For the Kinoshita–Terasaka knot we calculate

$$\mathcal{T}_{KT}(t) \approx \begin{array}{ll} (4.418 + 0.376i)t^3 & +(-22.942 - 4.845i)t^2 \\ + \cdots + (-22.926 - 4.845i)t^{-2} & +(4.418 + 0.376i)t^{-3}. \end{array}$$

This shows that $\mathcal{T}_K(t)$ detects mutation.The evaluations $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ conjecturally do *not* detect mutation. E.g. for both knots evaluation at $t = 1$ gives $4.1860 - 4.2286i$ and for $t = -1$ we get $261.34 + 102.13i$.

The Kinoshita-Terasaka knot and the Conway knot

The Conway knot and the Kinoshita-Terasaka knot are mutants knots with trivial Alexander polynomial.We calculate

$$\begin{aligned}\mathcal{T}_{Conway}(t) \approx \quad & (4.895 - 0.099i)t^5 \qquad +(-15.686 + 0.297i)t^4 \\ & + \dots (-15.686 + 0.298i)t^{-4} + (4.895 - 0.099i)t^{-5}.\end{aligned}$$

For the Kinoshita–Terasaka knot we calculate

$$\begin{aligned}\mathcal{T}_{KT}(t) \approx \quad & (4.418 + 0.376i)t^3 \qquad +(-22.942 - 4.845i)t^2 \\ & + \cdots + (-22.926 - 4.845i)t^{-2} \; +(4.418 + 0.376i)t^{-3}.\end{aligned}$$

This shows that $\mathcal{T}_K(t)$ detects mutation.The evaluations $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ conjecturally do *not* detect mutation. E.g. for both knots evaluation at $t = 1$ gives $4.1860 - 4.2286i$ and for $t = -1$ we get $261.34 + 102.13i$.

Questions. (1) Is $|\mathcal{T}_K(1)|$ always less than $|\mathcal{T}_K(-1)|$?

The Kinoshita-Terasaka knot and the Conway knot

The Conway knot and the Kinoshita-Terasaka knot are mutants knots with trivial Alexander polynomial.We calculate

$$\begin{aligned}\mathcal{T}_{Conway}(t) \approx \quad & (4.895 - 0.099i)t^5 & +(-15.686 + 0.297i)t^4 \\ & + \dots (-15.686 + 0.298i)t^{-4} & +(4.895 - 0.099i)t^{-5}.\end{aligned}$$

For the Kinoshita–Terasaka knot we calculate

$$\begin{aligned}\mathcal{T}_{KT}(t) \approx \quad & (4.418 + 0.376i)t^3 & +(-22.942 - 4.845i)t^2 \\ & + \dots + (-22.926 - 4.845i)t^{-2} & +(4.418 + 0.376i)t^{-3}.\end{aligned}$$

This shows that $\mathcal{T}_K(t)$ detects mutation.The evaluations $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ conjecturally do *not* detect mutation. E.g. for both knots evaluation at $t = 1$ gives $4.1860 - 4.2286i$ and for $t = -1$ we get $261.34 + 102.13i$.

Questions. (1) Is $|\mathcal{T}_K(1)|$ always less than $|\mathcal{T}_K(-1)|$?
(2) Does $\mathcal{T}_K(t)$ distinguish hyperbolic knots?
Added after talk: It does not, there are mutants with same $\mathcal{T}_K$.

$\mathcal{T}_K(t)$ and the knot genus

We now write $x(K) = 2\text{genus}(K) - 1$.

$\mathcal{T}_K(t)$ and the knot genus

We now write $x(K) = 2\text{genus}(K) - 1$.

Theorem (F-Taehee Kim) For any hyperbolic knot K we have

$$\deg(\mathcal{T}_K(t)) \leq 2x(K).$$

$\mathcal{T}_K(t)$ and the knot genus

We now write $x(K) = 2\text{genus}(K) - 1$.

Theorem (F-Taehee Kim) For any hyperbolic knot K we have

$$\deg(\mathcal{T}_K(t)) \leq 2x(K).$$

E.g. if K is the Conway knot, then the genus is three and we calculated

$$\deg(\mathcal{T}_{\text{Conway}}(t)) = 10.$$

$\mathcal{T}_K(t)$ and the knot genus

We now write $x(K) = 2\text{genus}(K) - 1$.

Theorem (F-Taehee Kim) For any hyperbolic knot K we have

$$\deg(\mathcal{T}_K(t)) \leq 2x(K).$$

E.g. if K is the Conway knot, then the genus is three and we calculated

$$\deg(\mathcal{T}_{\text{Conway}}(t)) = 10.$$

Similarly the genus of the Kinoshita-Terasaka knot is two and

$$\deg(\mathcal{T}_{\text{Kinoshita-Terasaka}}(t)) = 6.$$

$\mathcal{T}_K(t)$ and the knot genus

We now write $x(K) = 2\text{genus}(K) - 1$.

Theorem (F-Taehee Kim) For any hyperbolic knot K we have

$$\deg(\mathcal{T}_K(t)) \leq 2x(K).$$

E.g. if K is the Conway knot, then the genus is three and we calculated

$$\deg(\mathcal{T}_{\text{Conway}}(t)) = 10.$$

Similarly the genus of the Kinoshita-Terasaka knot is two and

$$\deg(\mathcal{T}_{\text{Kinoshita-Terasaka}}(t)) = 6.$$

We show that $\deg(\mathcal{T}_K(t)) = 2x(K)$ for all knots with at most thirteen crossings.

$\mathcal{T}_K(t)$ and the knot genus

We now write $x(K) = 2\text{genus}(K) - 1$.

Theorem (F-Taehee Kim) For any hyperbolic knot K we have

$$\deg(\mathcal{T}_K(t)) \leq 2x(K).$$

E.g. if K is the Conway knot, then the genus is three and we calculated

$$\deg(\mathcal{T}_{\text{Conway}}(t)) = 10.$$

Similarly the genus of the Kinoshita-Terasaka knot is two and

$$\deg(\mathcal{T}_{\text{Kinoshita-Terasaka}}(t)) = 6.$$

We show that $\deg(\mathcal{T}_K(t)) = 2x(K)$ for all knots with at most thirteen crossings. The genera of all thirteen crossings were also independently determined by Stoimenow.

$\mathcal{T}_K(t)$ and the knot genus

We now write $x(K) = 2\text{genus}(K) - 1$.

Theorem (F-Taehee Kim) For any hyperbolic knot K we have

$$\deg(\mathcal{T}_K(t)) \leq 2x(K).$$

E.g. if K is the Conway knot, then the genus is three and we calculated

$$\deg(\mathcal{T}_{\text{Conway}}(t)) = 10.$$

Similarly the genus of the Kinoshita-Terasaka knot is two and

$$\deg(\mathcal{T}_{\text{Kinoshita-Terasaka}}(t)) = 6.$$

We show that $\deg(\mathcal{T}_K(t)) = 2x(K)$ for all knots with at most thirteen crossings. The genera of all thirteen crossings were also independently determined by Stoimenow.

Conjecture. For any hyperbolic knot K we have

$$\deg(\mathcal{T}_K(t)) = 2x(K).$$

$\mathcal{T}_K(t)$ and fibered knots

Theorem (Goda-Kitano-Morifuji) If K is hyperbolic and fibered, then

$$\mathcal{T}_K(t) \text{ is monic and } \deg(\mathcal{T}_K(t)) = 2x(K).$$

$\mathcal{T}_K(t)$ and fibered knots

Theorem (Goda-Kitano-Morifuji) If K is hyperbolic and fibered, then

$$\mathcal{T}_K(t) \text{ is monic and } \deg(\mathcal{T}_K(t)) = 2x(K).$$

We show that $\mathcal{T}_K(t)$ detects all fibered knots with at most thirteen crossings.

$\mathcal{T}_K(t)$ and fibered knots

Theorem (Goda-Kitano-Morifuji) If K is hyperbolic and fibered, then

$$\mathcal{T}_K(t) \text{ is monic and } \deg(\mathcal{T}_K(t)) = 2x(K).$$

We show that $\mathcal{T}_K(t)$ detects all fibered knots with at most thirteen crossings.

Conjecture. A hyperbolic knot K is fibered if and only if $\mathcal{T}_K(t)$ is monic and if

$$\deg(\mathcal{T}_K(t)) = 2x(K)$$

holds.

$\mathcal{T}_K(t)$ and fibered knots

Theorem (Goda-Kitano-Morifuji) If K is hyperbolic and fibered, then

$$\mathcal{T}_K(t) \text{ is monic and } \deg(\mathcal{T}_K(t)) = 2x(K).$$

We show that $\mathcal{T}_K(t)$ detects all fibered knots with at most thirteen crossings.

Conjecture. A hyperbolic knot K is fibered if and only if $\mathcal{T}_K(t)$ is monic and if

$$\deg(\mathcal{T}_K(t)) = 2x(K)$$

holds.

Note that it is known that the set of twisted Alexander polynomials corresponding to *all* finite representations detects whether K is fibered.

The adjoint representation

Let K be a hyperbolic knot and $\alpha : \pi \to \mathrm{SL}(2, \mathbb{C})$ the canonical representation.

The adjoint representation

Let K be a hyperbolic knot and $\alpha : \pi \to \mathrm{SL}(2, \mathbb{C})$ the canonical representation. Then consider the *adjoint* representation

$$\alpha_{adj} : \pi \to \mathfrak{sl}(2, \mathbb{C}).$$

The adjoint representation

Let K be a hyperbolic knot and $\alpha : \pi \to \mathrm{SL}(2,\mathbb{C})$ the canonical representation.Then consider the *adjoint* representation

$$\alpha_{adj} : \pi \to \mathfrak{sl}(2,\mathbb{C}).$$

This is a 3-dimensional, irreducible and faithful representation.

The adjoint representation

Let K be a hyperbolic knot and $\alpha : \pi \to \mathrm{SL}(2, \mathbb{C})$ the canonical representation.Then consider the *adjoint* representation

$$\alpha_{adj} : \pi \to \mathfrak{sl}(2, \mathbb{C}).$$

This is a 3-dimensional, irreducible and faithful representation. In particular

$$\deg \tau(K, \alpha_{adj}) \leq \frac{1}{3} x(K).$$

The adjoint representation

Let K be a hyperbolic knot and $\alpha : \pi \to \mathrm{SL}(2, \mathbb{C})$ the canonical representation.Then consider the *adjoint* representation

$$\alpha_{adj} : \pi \to \mathfrak{sl}(2, \mathbb{C}).$$

This is a 3-dimensional, irreducible and faithful representation. In particular

$$\deg \tau(K, \alpha_{adj}) \leq \frac{1}{3} x(K).$$

For the Conway knot (which has genus 3) we compute

$$\deg \tau(\text{Conway knot}, \alpha_{adj}) = 13.$$

I.e. here the inequality is a strict inequality.

The character variety

The character variety is defined as

$$X(K) := \text{Hom}(\pi_1(S^3 \setminus K), \text{SL}(2, \mathbb{C}))//SL(2, \mathbb{C})\text{-conjugation.}$$

The character variety

The character variety is defined as

$$X(K) := \mathrm{Hom}(\pi_1(S^3 \setminus K), \mathrm{SL}(2,\mathbb{C}))//SL(2,\mathbb{C})\text{-conjugation.}$$

The Alexander polynomial defines a function

$$X(K) \to \mathbb{C}[t^{\pm 1}].$$

The character variety

The character variety is defined as

$$X(K) := \mathrm{Hom}(\pi_1(S^3 \setminus K), \mathrm{SL}(2,\mathbb{C}))//SL(2,\mathbb{C})\text{-conjugation}.$$

The Alexander polynomial defines a function

$$X(K) \to \mathbb{C}[t^{\pm 1}].$$

Theorem.

$$\{\chi \in X(K) \,|\, \deg(\tau(K,\chi)) = 2x(K)\} \text{ is Zariski open}$$

$$\{\chi \in X(K) \,|\, \tau(K,\chi) \text{ is monic }\} \text{ is Zariski closed}$$