Four-ball genus

Definition

The *four-ball genus* of a knot $K \subset S^3$ is

$$g_4(K) = \min\{g(\Sigma) \mid \Sigma \text{ smoothly embedded in } B^4 \text{ with } \partial \Sigma = K\}$$
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- Two knots $K_1$ and $K_2$ are *concordant* if

$$g_4(K_1 \# - K_2) = 0$$
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- The four-ball genus is bounded above by the Seifert genus $g(K)$ of a knot
  $$g_4(K) \leq g(K)$$
Recall:

- **Signature** of a knot, $\sigma(K)$, is the signature of $V + V^t$ where $V$ is a Seifert matrix for the knot.
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**Example**

For the trefoil knot $K = T_{2,3}$, $g(K) = 1$ and $\sigma(K) = \pm 2$. Therefore,

$$g_4(K) = 1.$$
A similarly-behaved invariant, $\tau(K)$, is constructed using Heegaard Floer theory.
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Heegaard Floer theory was developed by Ozsváth-Szabó and Rasmussen, based on counts of holomorphic curves in symplectic manifolds.

Ozsváth-Szabó, Rasmussen

$Y$ closed, oriented three-manifold, $HF^-(Y)$, $\hat{HF}(Y)$, $HF^+(Y)$ are modules over the power series ring $\mathbb{Z}[[U]]$ (variants of Heegaard Floer homology). Furthermore, when $K \subset Y$ is a knot, there is a knot invariant $\hat{HFK}(Y, K)$. 
Example (Milnor conjecture)

For the torus knot \(K = T_{m,n}\), its Seifert genus \(g(K) = \frac{(m-1)(n-1)}{2}\).
Ozsváth-Szabó showed that \(\tau(K) = \frac{(m-1)(n-1)}{2}\).
Therefore, \(g_4(K) = \frac{(m-1)(n-1)}{2}\).
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**Remark:**

- Sometimes, Milnor conjecture refers to the equivalent statement that the unknottting number, $u(K)$, of the torus knot $T_{m,n}$ is $\frac{(m-1)(n-1)}{2}$. 
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- First proved by Kronheimer-Mrowka using gauge theoretic methods.
- Rasmussen gave a purely combinatorial proof using Khovanov homology, by means of the $s$-invariant.
Refinement of $\tau$-invariant

In general, four-ball genus is hard to compute.

Hom and the speaker defined a concordance invariant $\nu^+$, which gives a better bound on the four-ball genus than $\tau$:

$$\tau(K) \leq \nu^+(K) \leq g_4(K)$$
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**Theorem (Hom-W)**

For any positive integer $p$, there exists a knot $K$ with $\tau(K) \geq 0$ and

$$\tau(K) + p \leq \nu^+(K) = g_4(K).$$
Here is a concrete example:

**Proposition (Hom-W)**

Let $K$ be the knot $T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. Then

$$\nu^+(K_{p,3p-1}) = g_4(K_{p,3p-1}) = \frac{p(3p-1)}{2} + 1.$$
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- One can compute

$$\tau(K_{p,3p-1}) = \frac{3p(p-1)}{2},$$

thus

$$\nu^+(K_{p,3p-1}) - \tau(K_{p,3p-1}) = p + 1.$$
One can also show

\[ |\sigma(K_{p,3p-1})| \leq |\sigma(K)| + |\sigma(T_{p,3p-1})| \leq 4 + (p - 1)(3p - 2) = 3p^2 - 5p + 6, \]
One can also show

\[ |\sigma(K_p, 3p-1)| \leq |\sigma(K)| + |\sigma(T_p, 3p-1)| \]
\[ \leq 4 + (p - 1)(3p - 2) \]
\[ = 3p^2 - 5p + 6, \]

Thus,

\[ |g_4(K_p, 3p-1) - \frac{1}{2}|\sigma(K_p, 3p-1)| \geq 2p - 2 \]

The knot signature cannot detect the four-ball genus of the knots either.
Let $\text{CFK}^\infty(K)$ denote the $\mathbb{Z} \oplus \mathbb{Z}$-filtered knot Floer complex of $K$.

Quotient complexes

$$A_k^+ = C\{\max\{i, j - k\} \geq 0\} \quad \text{and} \quad B^+ = C\{i \geq 0\}$$

where $i$ and $j$ refer to the two filtrations.

Subquotient complexes

$$\hat{A}_k = C\{\max\{i, j - k\} = 0\} \quad \text{and} \quad \hat{B} = C\{i = 0\} \cong \hat{CF}(S^3)$$

Map

$$\nu_k^+: A_k^+ \rightarrow B^+ \quad \text{and} \quad h_k^+: A_k^+ \rightarrow B^+$$

$$\hat{\nu}_k: \hat{A}_k \rightarrow \hat{B} \quad \text{and} \quad \hat{h}_k: \hat{A}_k \rightarrow \hat{B}.$$
Definition of $\tau$, $\nu$ and $\nu^+$

- $\tau(K) := \min\{ k \in \mathbb{Z} \mid \nu_k \text{ is nontrivial on homology} \}$, where $\nu_k : C\{i = 0, j \leq k\} \to \widehat{CF}(S^3)$ denotes inclusion.
- $\nu(K) = \min\{ k \in \mathbb{Z} \mid \widehat{v}_k : \hat{A}_k \to \widehat{CF}(S^3) \text{ is nontrivial on homology} \}$.
- $\nu^+(K) := \min\{ k \in \mathbb{Z} \mid \nu^+_k : A^+_k \to CF^+(S^3), \quad \nu^+_k(1) = 1 \}$

The relationship between the above three invariants

$$\tau(K) \leq \nu(K) \leq \nu^+(K)$$

follows from the commutative diagram

$$\begin{array}{ccc}
\hat{A}_k & \xrightarrow{j_A} & A^+_k \\
\downarrow \widehat{v}_k & & \downarrow \nu^+_k \\
\hat{B} & \xrightarrow{j_B} & B^+.
\end{array}$$
Figure: For the knot $T_{2,9} \# - T_{2,3;2,5}$, $\tau = 0$, $\nu = 1$, $\nu^+ = 2$
Properties of $\nu^+$

- When $K$ is quasi-alternating or strongly quasi-positive,
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- When $K$ is quasi-alternating or strongly quasi-positive,
  $$\nu^+(K) = \tau(K).$$

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- An equivalent invariant $\nu^-(K)$ defined using $CFK^-$ by Ozsváth-Szabó-Stipsicz.
Alternative characterization of $\nu^+$

Recall the definition

$$\nu^+(K) := \min\{k \in \mathbb{Z} \mid \nu_k^+ : A_k^+ \to CF^+(S^3), \ \nu_k^+(1) = 1\}$$
Alternative characterization of $\nu^+$

Recall the definition

$$\nu^+(K) := \min\{k \in \mathbb{Z} | \nu_k^+ : A_k^+ \to CF^+(S^3), \; \nu_k^+(1) = 1\}$$

Let $V_k$ be the $U$-exponent of $\nu_k^+ : A_k^+ \to CF^+(S^3)$ at sufficient high gradings. Then, $\nu^+(K)$ is equivalent to the smallest $k$ such that $V_k = 0$. 

Question

Why is this alternative definition useful?

Answer: Because it is related to knot surgery.

Proposition (Ni-W)

Suppose $p, \; q > 0$, and fix $0 \leq i \leq p - 1$. Then

$$d(S^3/p, q(K), i) = d(L(p, q), i) - 2 \max\{V\lfloor i/q \rfloor, H\lfloor i - p/q \rfloor\}.$$
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Suppose $p, q > 0$, and fix $0 \leq i \leq p - 1$. Then

$$d(S^3_{p/q}(K), i) = d(L(p, q), i) - 2 \max\{V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}\}.$$
Fact: Cable knots has a reducible surgery

\[ S^3_{pq}(K_{p,q}) \cong S^3_{q/p}(K) \# L(p, q). \]
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Compare the correction terms of the two sides, one obtains

\[
d(L(pq, 1), i) - 2V_i(K_{p,q}) = d(L(q, p), p_1(i)) - 2 \max \{ V_{\left\lfloor \frac{p_1(i)}{p} \right\rfloor}(K), H_{\left\lfloor \frac{p_1(i) - q}{p} \right\rfloor}(K) \} + d(L(p, q), p_2(i)).
\]
Fact: Cable knots has a reducible surgery

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\]

When \( K \) is the unknot, the identity reduces to

\[
d(L(pq, 1), i) - 2V_i(T_{p,q}) = d(L(q, p), p_1(i)) + d(L(p, q), p_2(i)).
\]
Hence,

\[ V_i(K_{p,q}) = V_i(T_{p,q}) + \max\{ V\left\lfloor \frac{p_1(i)}{p} \right\rfloor(K), H\left\lfloor \frac{p_1(i) - q}{p} \right\rfloor(K) \} \]

\[ \geq \max\{ V\left\lfloor \frac{p_1(i)}{p} \right\rfloor(K), H\left\lfloor \frac{p_1(i) - q}{p} \right\rfloor(K) \} \]
Hence,

\[ V_i(K_p,q) = V_i(T_p,q) + \max\{ V_{\lfloor \frac{p_1(i)}{p} \rfloor}(K), H_{\lfloor \frac{p_1(i) - q}{p} \rfloor}(K) \} \]

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Using the alternative characterization of \( \nu^+(K) \) (the smallest \( k \) such that \( V_k = 0 \)), one concludes

**Theorem (Hom-W)**

Let \( K \) be a knot with \( \nu^+(K) = g_4(K) = n \), then

\[ \nu^+(K_p,(2n-1)p-1) = g_4(K_p,(2n-1)p-1) = \frac{p((2n-1)p-1)}{2} + 1. \]
In particular, the knot $K = T_{2,5} \# 2 T_{2,3} \# - T_{2,3;2,5}$ has

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It follows:

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Let $K$ be the knot $T_{2,5} \# 2 T_{2,3} \# - T_{2,3;2,5}$. Then

$$\nu^+(K_p, 3p-1) = g_4(K_p, 3p-1) = \frac{p(3p - 1)}{2} + 1.$$
Thank you very much!