Unknotting operation for fibered knots and pseudo-fiber surfaces.

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In this talk we treat knots and links in $S^3$.

link:

knot:

trivial knot

trefoil knot

figure eight knot
Definition (unknotting operation)

$L$: link

$D$: a disk in $S^3$ which intersects $L$ in two points with different orientations.

$L':$ the link obtained from $L$ by applying $+1$ or $-1$ surgery along $\partial D$.

We say that $L'$ is obtained from $L$ by an unknotting operation.

$K$: knot

The minimum of the number of unknotting operations required for transforming $K$ into a trivial knot is the unknotting number of $K$ (denoted by $u(K)$).
Introduction
Preliminaries
pseudo-fiber surface of level $n$
Application to unknotting number of fibered knots

(1937)
Wendtz

(1970’s)
Murasugi, Stallings

(1980’s)
Gabai $\rightarrow$ Sharlemann-Thompson, Kobayashi
Culler-Gordon-Luecke-Shalen $\rightarrow$ Kanenobu-Murakami

(1990’s)
Kronheimer-Mrowka $\rightarrow$ Rudolph, Tanaka

(2000’s)
Ozsváth-Szabó
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Definition (minimal genus Seifert surface, genus of a knot)

The genus of a knot $K$ is 

$$g(K) := \min \{\text{genus}(F) | F: \text{Seifert surface for } K\}$$

A Seifert surface $S$ for $K$ is 

minimal genus if $\text{genus}(S) = g(K)$.

Note. $K$: trivial knot $\iff g(K) = 0$
Theorem (Scharlemann-Thompson)

$L, L'$: knot
$L \rightarrow L'$: unknotting operation
$g(L') < g(L)$
$\exists S$: minimal genus Seifert surface of $L$
\hspace{1cm} s.t. $S$ is a plumbing of a surface and Hopf band,

and

\begin{center}
\includegraphics[width=0.5\textwidth]{image}
\end{center}

The minimal genus Seifert surfaces for $K$ are not necessarily unique (up to isotopy relative $K$).

**Fact**

For each fibered link (: defined later) $K$ the minimal genus Seifert surface are unique up to isotopy relative $K$.

(It is a *fiber surface*)
Corollary

Suppose $K$: fibered knot s.t. $u(K) = 1$. Then,

$\exists$ fiber surface $S$ for $K$.

s.t. $K \rightarrow K'$: trivial knot

Then $S'$: a Seifert surface for trivial knot
In fact, $S'$ is always what is called a pre-fiber surface (defined later). Moreover, for each $g(\geq 1)$, genus $g$ pre-fiber surface for the trivial knot is isotopic to

It is also studied that what kind of twists on $\Sigma^1_g$ produce fiber surfaces.

Motivation:

Generalize the above results for fibered knots with unknotting numbers $> 1$. 
$L$ : link
$S$ : Seifert surface for $L$
$E(L)(:= S^3 \setminus \text{Int} \text{N}(L))$ : exterior of $L$
(For simplicity we denote $S \cap E(L)$ by $S$.)
$N = N(S; E(L))$
$\delta = N \cap \partial E(L)$

$R_-(\delta)$ ($R_+(\delta)$) corresponds to $S \times \{0\}$ ($S \times \{1\}$).

The product sutured manifold $(N, \delta)$ is called the sutured manifold obtained from $S$. 
Definition (complementary sutured manifold)

\[ N^c = \text{cl}(E(L) \setminus N) \]
\[ \delta^c = \text{cl}(\partial E(L) \setminus \delta) \]
\[ R_{\pm}(\delta^c) = R_{\mp}(\delta) \]

\((N^c, \delta^c)\) is called the complementary sutured manifold for \(S\).
Let $L$ be a link, $S$ be a Seifert surface for $L$.

**Definition (fiber surface)**

$S$ is a *fiber surface*, if:

- the complementary sutured manifold is a product sutured manifold.

**Definition (fibered link, knot)**

$L$ is a *fibered link* (*knot*) if

\[ \exists S : \text{Seifert surface for } L \text{ s.t. } S \text{ is a fiber surface.} \]

\( S \): Seifert surface for a link \( L \)

\((N^c, \delta^c)\): complementary sutured manifold for \( S \)

**Definition (pre-fiber surface)**

\( S \) is a **pre-fiber surface**, if:

\[ \exists D^\pm (\subset N^c): \text{mutually disjoint disks} \]

s.t. \( R_\pm(\delta^c) \cap D^\pm = \partial D^\pm: \text{essential in } R_\pm(\delta^c) \)

(such disk is called a compressing disk for \( R_\pm(\delta^c) \)),

and \((N^{c'}, \delta^c)\) is a product sutured manifold,

where \( N^{c'} \) is a 3-manifold obtained from \( N^c \)

by cutting along \( D^+ \cup D^- \).

The compressing disks \( \tilde{D}^+, \tilde{D}^- \) for \( S \), which are corresponding to \( D^+, D^- \) are called **canonical compressing disks** for \( S \).
Example

Unknotting operation for fibered knots and pseudo-fiber surfaces.
Hirasawa’s construction of pre-fiber surface

Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_p$: sequence of signs $\pm$,

\[ \begin{array}{c}
\begin{array}{c}
\text{Notation:}
\end{array}
\end{array} \]

\[ D(\epsilon_1, \epsilon_2, \ldots, \epsilon_p): \text{the diagram given by:} \]

(Note: $D(\epsilon_1, \epsilon_2, \ldots, \epsilon_p)$ is isotopic to $D(\epsilon_2, \ldots, \epsilon_p, \epsilon_1)$ on $S^2$)

Let

\[ S(\epsilon_1, \epsilon_2, \ldots, \epsilon_p): \text{the Seifert surface obtained from } D(\epsilon_1, \epsilon_2, \ldots, \epsilon_p) \]

Hirasawa showed that

For each $n \geq 1$, \( S(\underbrace{+ \cdots + - \cdots -}_{n \quad n+1}) \) is isotopic to $\Sigma_n^1$, and \( S(\underbrace{+ \cdots + - \cdots -}_{n \quad n}) \) is also a pre-fiber surface.
Theorem (T. Kobayashi)

Suppose \( K: \) fibered knot s.t. \( u(K) = 1 \).

Then,

\( \exists \) fiber surface \( S \) for \( K \).

s.t.

\( K \rightarrow K': \) trivial knot

Then

\( S: \) fiber surface

\( S': \) pre-fiber surface
Theorem (T. Kobayashi)

$S$: pre-fiber surface with canonical compressing disks $\bar{D}^+, \bar{D}^-$. 
$\alpha$: an arc properly embedded in $S$
\[\text{s.t. } \alpha \cap \bar{D}^+: \text{ 1-point},\]
\[\text{and } \alpha \cap \bar{D}^-: \text{ 1-point}.\]

The surface obtained from $S$ by applying a twist along $\alpha$ is a fiber surface.

Example

Unknotting operation for fibered knots and pseudo-fiber surfaces.
Motivation:

Generalize the above results for fibered knots with unknotting numbers > 1.
A generalization of pre-fiber surface.

Let $L$: link
$S$: Seifert surface
$(N^c, \delta^c)$: complementary sutured manifold for $S$

Definition (pseudo-fiber surface of level $n$)

For $n \geq 0$, $S$ is a pseudo-fiber surface of level $n$, if:

$\exists D_1^\pm, \ldots, D_n^\pm$
: mutually disjoint compressing disks of $R_\pm(\delta^c)$

s.t. $(N^{c'}, \delta^c)$ is a product sutured manifold,
where $N^{c'}$ is a 3-manifold obtained from $N^c$ by cutting along
$D_1^+ \cup \cdots \cup D_n^+ \cup D_1^- \cup \cdots \cup D_n^-$

$S$ is a pseudo-fiber surface of level 0 $\iff$ $S$ is a fiber surface
$S$ is a pseudo-fiber surface of level 1 $\iff$ $S$ is a pre-fiber surface
Recall $\epsilon_1, \epsilon_2, \ldots, \epsilon_p$: sequence of signs $\pm$

(We consider the cyclic order of it.)

$D(\epsilon_1, \epsilon_2, \ldots, \epsilon_p)$: the diagram given by:

$S(\epsilon_1, \epsilon_2, \ldots, \epsilon_p)$: the Seifert surface obtained from $D(\epsilon_1, \epsilon_2, \ldots, \epsilon_p)$

We can decompose it up to cyclic permutation into blocks $B_1, B_2, \ldots, B_m$ s.t. $(-\ldots-+\ldots+\ldots+\ldots+)$

Note: If $(\epsilon_1, \epsilon_2, \ldots, \epsilon_p) \neq (+, \ldots, +)$ or $(-, \ldots, -)$ then $m$ is an even number.
**Proposition**

Suppose \((\epsilon_1, \epsilon_2, \ldots, \epsilon_p) \neq (+, \ldots, +) \text{ or } (-, \ldots, -),\)

(We can decompose it into blocks \(B_1, B_2, \ldots, B_m.\))

Then \(S(\epsilon_1, \epsilon_2, \ldots, \epsilon_p)\) is a pseudo-fiber surface of level \(\frac{m}{2}.\)

Moreover a system of canonical compressing disks for \(S(\epsilon_1, \epsilon_2, \ldots, \epsilon_p)\) appears at each pair of \(-+\), or \(+-\) as in.
A generalization of Theorem (T. Kobayashi)

**Theorem (T. Kobayashi)**

*\(S\): pre-fiber surface with canonical compressing disks \(\bar{D}^+, \bar{D}^-\).

*\(\alpha\): arcs properly embedded in \(S\) s.t. \(\alpha \cap \bar{D}^+\): 1-point, and \(\alpha \cap \bar{D}^-\): 1-point.

The surface obtained from \(S\) by applying a twist along \(\alpha\) is a fiber surface.

**Theorem 4.1**

Let \(S\): a pseudo-fiber surface of level \(n\)\(^1\)

*\(\alpha_1\), \ldots, \(\alpha_p\): mutually disjoint arcs properly embedded in \(S\)

Suppose that \(\exists \bar{D}_1^+, \ldots, \bar{D}_n^+, \bar{D}_1^- , \ldots, \bar{D}_n^-\) \((p \leq n)\)

: a system of canonical compressing disks

s.t. \(\alpha_i \cap \partial \bar{D}_i^+\) (\(\partial \bar{D}_i^-\) resp.): 1-point. \((i = 1, \ldots, p)\),

\(\alpha_i \cap \bar{D}_j^\pm = \emptyset\) for \(\forall i, j\) \((i \neq j)\)

Then

any surface obtained from \(S\) by twisting along \(\alpha_1 \cup \cdots \cup \alpha_p\)

is a pseudo-fiber surface of level \(n - p\).
Example 4.2

Unknotting operation for fibered knots and pseudo-fiber surfaces.
Theorem (T. Kobayashi)

Suppose $K$: fibered knot s.t. $u(K) = 1$.
Then, there exists a fiber surface $S$ for $K$.

s.t. $K$: fibered $K'$: trivial knot

Then $S'$: pre-fiber surface
Definition (Ascending sequence of pseudo-fiber surfaces)

A sequence of pseudo-fiber surfaces $S_i$ of level $p_i$

$$S_0 \rightarrow S_1 \rightarrow \cdots \rightarrow S_i \rightarrow S_{i+1} \rightarrow \cdots \rightarrow S_n$$

is an ascending sequence of pseudo-fiber surfaces if:

- $p_0 = 0$, $p_i \leq p_{i+1}$ ($i = 0, 1, \ldots, n - 1$)
- $\exists \alpha_1^{(i)}, \ldots, \alpha_{q_i}^{(i)}$: mutually disjoint arcs properly embedded in $S_i$

such that $S_{i+1}$ is obtained from $S_i$ by twisting along $\alpha_1^{(i)} \cup \cdots \cup \alpha_{q_i}^{(i)}$. 

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Unknotting operation for fibered knots and pseudo-fiber surfaces
Question

For each fibered knot $K$, does there exist an ascending sequence of pseudo-fiber surfaces realizing the unknotting number $u(K)$?

More precisely, does there exist an ascending sequence of pseudo-fiber surfaces

$$S_0 \to S_1 \to \cdots \to S_n$$

(with levels $p_i$, arcs $\alpha_1^{(i)}, \ldots, \alpha_{q_i}^{(i)}$ as above) such that $\partial S_0 = K$, $\partial S_n =$ trivial knot, and $u(K) = q_0 + q_1 + \cdots + q_{n-1}$?
In [Fu], I gave an affirmative answer to Question for torus knot.

Fibered knots with crossing number $\leq 8$

Example

\[ u(3_1) = 1 \]
Fibered knots with crossing number $\leq 8$

Example

$u(9_{31}) = 2$
Fibered knots with crossing number $\leq 8$

Example

$u(8_{16}) = 2$
Fibered knots with crossing number $\leq 8$

Example

$\Sigma_0$

$\Sigma_1$

$\Sigma_2$

$u(8_5) = 2$
Fibered knots with crossing number $\leq 8$