Stein fillings of contact 3-manifolds obtained as Legendrian surgeries

Youlin Li
(With Amey Kaloti)
Shanghai Jiao Tong University

A Satellite Conference of Seoul ICM 2014
Knots and Low Dimensional Manifolds
August 23, 2014
Suppose $M$ is an oriented connected 3-dim smooth manifold, and $\alpha \in \Omega^1(M)$. If $\alpha \wedge d\alpha > 0$, then $\alpha$ is called a contact form, and $\xi = \ker \alpha$ is called a contact structure.
Suppose $M$ is an oriented connected 3-dim smooth manifold, and $\alpha \in \Omega^1(M)$. If $\alpha \wedge d\alpha > 0$, then $\alpha$ is called a contact form, and $\xi = \ker \alpha$ is called a contact structure.

Equivalently, a contact structure $\xi$ on $M$ is a nowhere integrable plane field in $TM$. 
Example

Suppose the 3-space $\mathbb{R}^3$ has coordinate $(x, y, z)$, then $\alpha = dz - ydx$ is a contact form, since $\alpha \wedge d\alpha = dx \wedge dy \wedge dz \neq 0$. $\ker \alpha = \langle \partial_y, \partial_x + y\partial_z \rangle$ is a contact structure on $\mathbb{R}^3$. We denote it by $(\mathbb{R}^3, \xi_{st})$. 
Example
Suppose the 3-space $\mathbb{R}^3$ has coordinate $(x, y, z)$, then 
$\alpha = dz - ydx$ is a contact form, since $\alpha \wedge d\alpha = dx \wedge dy \wedge dz \neq 0$. 
$\ker \alpha = \langle \partial_y, \partial_x + y\partial_z \rangle$ is a contact structure on $\mathbb{R}^3$. We denote it by $(\mathbb{R}^3, \xi_{st})$. 

Youlin Li (With Amey Kaloti)
Stein fillings of contact 3-manifolds obtained as Legendrian Surgery
Examples of contact structures

Example

Suppose $S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \sum_{i=1}^{4} x_i^2 = 1\}$, then it has a contact structure $\ker(x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3)$. We denote it by $(S^3, \xi_{st})$.

Suppose $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$, then $\xi_{st} = TS^3 \cap J(TS^3)$, where $J$ is an almost complex structure on $\mathbb{C}^2$.
Examples of contact structures

**Example**

Suppose $S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \sum_{i=1}^{4} x_i^2 = 1\}$, then it has a contact structure $\ker(x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3)$. We denote it by $(S^3, \xi_{st})$.

Suppose $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$, then $\xi_{st} = TS^3 \cap J(TS^3)$, where $J$ is an almost complex structure on $\mathbb{C}^2$.

**Remark:** $\forall$ point $pt \in S^3$, $(S^3 \setminus \{pt\}, \xi_{st}|_{S^3 \setminus \{pt\}})$ is contactomorphic to $(\mathbb{R}^3, \xi_{st})$. 

Youlin Li (With Amey Kaloti)
Legendrian knots

Given a contact 3-manifold \((M, \xi)\), a knot \(L \subset M\) is called **Legendrian** if \(L\) is everywhere tangent to \(\xi\), i.e., \(L'(p) \in \xi_p\) at every point \(p \in L\).
Legendrian knots

Given a contact 3-manifold \((M, \xi)\), a knot \(L \subset M\) is called \textbf{Legendrian} if \(L\) is everywhere tangent to \(\xi\), i.e., \(L'(p) \in \xi_p\) at every point \(p \in L\).

Two Legendrian knots \(L_0\) and \(L_1\) are \textbf{Legendrian isotopic} if there is a continuous family \(L_t, t \in [0, 1]\), of Legendrian knots starting at \(L_0\) and ending at \(L_1\).
Legendrian knots

Given a contact 3-manifold \((M, \xi)\), a knot \(L \subset M\) is called \textbf{Legendrian} if \(L\) is everywhere tangent to \(\xi\), i.e., \(L'(p) \in \xi_p\) at every point \(p \in L\).

Two Legendrian knots \(L_0\) and \(L_1\) are \textbf{Legendrian isotopic} if there is a continuous family \(L_t, t \in [0, 1]\), of Legendrian knots starting at \(L_0\) and ending at \(L_1\).

In this talk, we focus on the Legendrian knots in \((S^3, \xi_{st})\), which can be viewed clearly in \((\mathbb{R}^3, \xi_{st}) \hookrightarrow (S^3, \xi_{st})\).
The projection of a Legendrian knot in \((\mathbb{R}^3, \xi_{st} = \ker(dz - ydx))\) on the \(xz\)-plane is the **front projection**.
If a knot $K \subset S^3$ is contained in a surface $F$, then the **surface framing** of $K$ is a vector field which spans $T_pF$ with $K'(p)$ at every point $p \in K$.

In particular, if $F$ is the Seifert surface of $K$, then we call the surface framing as **Seifert framing**.
If a knot $K \subset S^3$ is contained in a surface $F$, then the **surface framing** of $K$ is a vector field which spans $T_pF$ with $K'(p)$ at every point $p \in K$.

In particular, if $F$ is the Seifert surface of $K$, then we call the surface framing as **Seifert framing**.

The **contact framing** of a Legendrian knot $L \subset (M, \xi)$ is a vector field which spans $\xi_p$ with $L'(p)$ at every point $p \in L$. 

Youlin Li (With Amey Kaloti) 

Stein fillings of contact 3-manifolds obtained as Legendrian sur
If a knot $K \subset S^3$ is contained in a surface $F$, then the **surface framing** of $K$ is a vector field which spans $T_pF$ with $K'(p)$ at every point $p \in K$.

In particular, If $F$ is the Seifert surface of $K$, then we call the surface framing as **Seifert framing**.

The **contact framing** of a Legendrian knot $L \subset (M, \xi)$ is a vector field which spans $\xi_p$ with $L'(p)$ at every point $p \in L$.

The **Thurston-Bennequin invariant** of a Legendrian knot $L \subset (S^3, \xi_{st})$ measures the contact framing with respect to the Seifert framing.
Suppose $L$ is a Legendrian knot in $(S^3, \xi_{st})$, and $F$ is a Seifert surface of $L$, then $\xi_{st}|_F$ form a trivial two dimensional bundle. A trivialization of $\xi_{st}|_F$ induces a trivialization $\xi_{st}|_L = L \times \mathbb{R}^2$.

Let $v$ be the tangent vector field of $L$. The winding number of $v$ along $L$ w.r.t. the trivialization $\xi_{st}|_L = L \times \mathbb{R}^2$ is called the rotation number of $L$, denoted by $\text{rot}(L)$. 

Youlin Li (With Amey Kaloti)
We define the local operation **positive (negative) stabilization**, by adding a downward (upward) zigzag in the front projection.
We define the local operation **positive (negative) stabilization**, by adding a downward (upward) zigzag in the front projection.

1. \( S_+ S_- (L) = S_- S_+(L) \).
We define the local operation **positive (negative) stabilization**, by adding a downward (upward) zigzag in the front projection.

1. \( S_+ S_-(L) = S_- S_+(L) \).
2. \( \text{rot}(S_\pm(L)) = \text{rot}(L) \pm 1 \).
We define the local operation **positive (negative) stabilization**, by adding a downward (upward) zigzag in the front projection.

1. $S_+ S_-(L) = S_- S_+(L)$.
2. $\text{rot}(S_\pm(L)) = \text{rot}(L) \pm 1$.
3. $\text{tb}(S_\pm(L)) = \text{tb}(L) - 1$.

Youlin Li (With Amey Kaloti)
Classification of Legendrian twist knots in \((S^3, \xi_{st})\)

**Twist knot**, denoted by \(K_m\), is the knot of the following type.

The box contains \(m\) right handed half twists if \(m \geq 0\), and \(-m\) left handed half twists if \(m < 0\).
Classification of Legendrian twist knots in \((S^3, \xi_{st})\)

**Twist knot**, denoted by \(K_m\), is the knot of the following type.

The box contains \(m\) right handed half twists if \(m \geq 0\), and \(-m\) left handed half twists if \(m < 0\).

\(K_{-2}\) *trefoil*  \(K_2\) *figure-of-eight*  \(K_4\) *stevedore's*
Theorem (Etnyre-Ng-Vértesi)

For any integer $m$, the Legendrian twist knot $K_m$ can be classified up to Legendrian isotopy.

In particular, for $p \geq 1$, there is a unique Legendrian twist knot $K_{-2p}$, denoted by $L_0$, with $tb = -1$ and $rot = 0$.

If a Legendrian twist knot $K_{-2p}$ has the same $tb$ and $rot$ with some stabilization of $L_0$, then it is Legendrian isotopic to that stabilization of $L_0$. 
Classification of Legendrian twist knots

This is the Legendrian twist knot $K_{-2p}, L_0$, with $tb = -1$ and $rot = 0$.

The box contains $2p - 2$ “S”-shape Legendrian tangle.
This is a stabilization of $L$, $S_{n-k}^n S_{k-1}^k(L_0)$, where $n \geq k \geq 1$. 
A 2-dimensional complex manifold $S$ is a **Stein surface** if it admits a proper biholomorphic embedding $S \hookrightarrow \mathbb{C}^4$. 

Example: $D^4 \subset \mathbb{C}^2$. Let $M = \partial S$, then $\xi = \ker(TM) \cap J(TM)$ is a contact structure on $M$. We call the Stein domain $S_c$ a Stein filling of $(M, \xi)$. 

Example: $D^4 \subset \mathbb{C}^2$ is a Stein filling of $(S^3, \xi_{st})$. 

Youlin Li (With Amey Kaloti)
A 2-dimensional complex manifold $S$ is a **Stein surface** if it admits a proper biholomorphic embedding $S \hookrightarrow \mathbb{C}^4$.

[Grauert] A 2-dimensional complex manifold $S$ is Stein iff it admits a proper exhausting plurisubharmonic function $\rho : S \to [0, \infty)$. 

Example: $D_4 \subset \mathbb{C}^2$.

Let $M = \partial S_c$, then $\xi = TM \cap J(TM)$ is a contact structure on $M$.

We call the Stein domain $S_c$ a **Stein filling** of $(M, \xi)$.

Example: $D_4 \subset \mathbb{C}^2$ is a Stein filling of $(S_3, \xi_{st})$.
A 2-dimensional complex manifold $S$ is a **Stein surface** if it admits a proper biholomorphic embedding $S \hookrightarrow \mathbb{C}^4$.

[Grauert] A 2-dimensional complex manifold $S$ is Stein iff it admits a proper exhausting plurisubharmonic function $\rho : S \rightarrow [0, \infty)$. For any regular value $c$ of $\rho$, the manifold $S_c := \{ p \in S : \rho(p) \leq c \}$ is a **Stein domain**.

**Example:** $D^4 \subset \mathbb{C}^2$. 

Youlin Li (With Amey Kaloti)  
Stein fillings of contact 3-manifolds obtained as Legendrian sur
A 2-dimensional complex manifold $S$ is a **Stein surface** if it admits a proper biholomorphic embedding $S \hookrightarrow \mathbb{C}^4$.

[Grauert] A 2-dimensional complex manifold $S$ is Stein iff it admits a proper exhausting plurisubharmonic function $\rho : S \to [0, \infty)$.  

For any regular value $c$ of $\rho$, the manifold $S_c := \{ p \in S : \rho(p) \leq c \}$ is a **Stein domain**.  

*Example:* $D^4 \subset \mathbb{C}^2$.  

Let $M = \partial S_c$, then $\xi = TM \cap J(TM)$ is a contact structure on $M$.  
We call the Stein domain $S_c$ a **Stein filling** of $(M, \xi)$.  
*Example:* $D^4 \subset \mathbb{C}^2$ is a Stein filling of $(S^3, \xi_{st})$.  

Youlin Li (With Amey Kaloti)  
Stein fillings of contact 3-manifolds obtained as Legendrian surgeries
Given a Legendrian knot $L$ in any contact 3-manifold $(M, \xi)$, a **Legendrian surgery** on $L$ yields the contact 3-manifold $(M', \xi')$, where

1. $M'$ is obtained from $M$ by a Dehn surgery on $L$ with framing $-1$ rel. to the contact framing, and
2. $\xi'$ is obtained by extending $\xi|_{M\setminus N(L)}$ to the reglued copy of $N(L)$, where $N(L)$ is the standard convex nbhd of $L$ in $(M, \xi)$. 

Youlin Li (With Amey Kaloti)
The **Weinstein handle** $H$ is a 4-dimensional 2-handle

$$D^2 \times D^2 = \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 : x_1^2 + x_2^2 \leq 1, y_1^2 + y_2^2 \leq 1\}$$

with the symplectic form $d\lambda$, where

$$\lambda = 1.5(x_1 dy_1 + x_2 dy_2) + 0.5(y_1 dx_1 + y_2 dx_2).$$

Both $S^1 \times D^2$ and $D^2 \times S^1$ are contact solid tori. The attaching sphere $S$ is Legendrian in $S^1 \times D^2$. 

[Diagram of Weinstein handle]
Theorem (Eliashberg)

Suppose that $W$ is a (complex) 2-dimensional Stein domain and $L \subset \partial W$ is a Legendrian knot. By attaching a Weinstein handle $H$ to $W$ along $L$ with framing $-1$ rel. to its contact framing to $\partial W$ along $L$, the Stein structure can be extended uniquely to $W' = W \cup H$. 
Theorem (Eliashberg)

Suppose that $W$ is a (complex) 2-dimensional Stein domain and $L \subset \partial W$ is a Legendrian knot. By attaching a Weinstein handle $H$ to $W$ along $L$ with framing $-1$ rel. to its contact framing to $\partial W$ along $L$, the Stein structure can be extended uniquely to $W' = W \cup H$.

Corollary

A Legendrian link $L \subset (S^3, \xi_{st})$ determines a Stein domain $X_L$. Topologically $X_L$ is given by 2-handle attachments along the link $L$ with framings $-1$ rel. to the contact framings of the individual components. In particular, the Legendrian surgery on $(S^3, \xi_{st})$ along $L$ yields a contact 3-manifold which has a Stein filling $X_L$. 

Youlin Li (With Amey Kaloti)

Stein fillings of contact 3-manifolds obtained as Legendrian sur...
Stein fillings of a given contact 3-manifold are classified up to homeomorphism, up to diffeomorphism, up to Symplectic deformation, or up to symplectomorphism.
Classification of Stein fillings

Stein fillings of a given contact 3-manifold are classified up to homeomorphism, up to diffeomorphism, up to Symplectic deformation, or up to symplectomorphism.

Symplectomorphism $\Rightarrow$ Symplectic deformation $\Rightarrow$ diffeomorphism $\Rightarrow$ homeomorphism
Classification of Stein fillings

Stein fillings of a given contact 3-manifold are classified up to homeomorphism, up to diffeomorphism, up to Symplectic deformation, or up to symplectomorphism.

Symplectomorphism $\Rightarrow$ Symplectic deformation $\Rightarrow$ diffeomorphism $\Rightarrow$ homeomorphism

[Elisashberg] There is a unique Stein filling, up to symplectic deformation, of $(S^3, \xi_{st})$. 
[Stipsicz] There is a unique Stein filling, up to homeomorphism, of the Poincaré homology sphere $\Sigma(2, 3, 5)$ and the 3-torus $T^3$. 
[Stipsicz] There is a unique Stein filling, up to homeomorphism, of the Poincaré homology sphere $\Sigma(2, 3, 5)$ and the 3-torus $T^3$.

[Wendl] There is a unique Stein filling, up to symplectomorphism, of the 3-torus $T^3$. 

Remark: One of the two above Stein fillings of $L(4, 1)$ can be obtained by attaching Weinstein handle.
[Stipsicz] There is a unique Stein filling, up to homeomorphism, of the Poincaré homology sphere $\Sigma(2, 3, 5)$ and the 3-torus $T^3$.

[Wendl] There is a unique Stein filling, up to symplectomorphism, of the 3-torus $T^3$.

[McDuff] For a universally tight contact structure, there is a unique Stein filling of a lens space $L(p, 1)$ ($p \neq 4$), and there are two Stein fillings of the lens space $L(4, 1)$, up to diffeomorphism.

Remark: One of the two above Stein fillings of $L(4, 1)$ can be obtained by attaching Weinstein handle.
[Plamenevskaya-Van Horn-Morris] There is a unique Stein filling, up to symplectic deformation, of $L(p, 1)$ with any virtually overtwisted (i.e., not universally tight) tight contact structure.
[Plamenevskaya-Van Horn-Morris] There is a unique Stein filling, up to symplectic deformation, of $L(p, 1)$ with any virtually overtwisted (i.e., not universally tight) tight contact structure.

Remark: They used the fact that these contact 3-manifolds are obtained by Legendrian surgeries on Legendrian unknots in $(S^3, \xi_{st})$ with mixed stabilizations.
[Lisca] classified the Stein fillings of universally tight lens spaces $L(p, q)$ up to diffeomorphism.
[Lisca] classified the Stein fillings of universally tight lens spaces $L(p, q)$ up to diffeomorphism.

[Kaloti] classified the Stein fillings of virtually overtwisted tight lens spaces $L(pm + 1, m)$ up to symplectic deformation.
[Lisca] classified the Stein fillings of universally tight lens spaces $L(p, q)$ up to diffeomorphism.

[Kaloti] classified the Stein fillings of virtually overtwisted tight lens spaces $L(pm + 1, m)$ up to symplectic deformation.

[Ohta-Ono] classified the Stein fillings of some links of simple singularities up to diffeomorphism.
[Lisca] classified the Stein fillings of universally tight lens spaces $L(p, q)$ up to diffeomorphism.

[Kaloti] classified the Stein fillings of virtually overtwisted tight lens spaces $L(pm + 1, m)$ up to symplectic deformation.

[Ohta-Ono] classified the Stein fillings of some links of simple singularities up to diffeomorphism.

[Starkston] gave finiteness results and some classifications, up to diffeomorphism, of minimal Stein fillings of certain contact Seifert fibered spaces over $S^2$.

......
Main result I

**Theorem (Kaloti-L.)**

*If* $L_0$ *is a Legendrian twist knot* $K_{-2p}$ *with* $tb = -1$ *and* $rot = 0$, *then the Legendrian surgery on* $(S^3, \xi_{st})$ *along any stabilization of* $L_0$ *yields a contact 3-manifold with unique Stein filling up to symplectic deformation.*
Main result I

**Theorem (Kaloti-L.)**

If $L_0$ is a Legendrian twist knot $K_{-2p}$ with $tb = -1$ and $rot = 0$, then the Legendrian surgery on $(S^3, \xi_{st})$ along any stabilization of $L_0$ yields a contact 3-manifold with unique Stein filling up to symplectic deformation.

**Remark:** If the stabilization on $L_0$ is $S^{n-k}_+ S^{k-1}_-(L_0)$, then the resulted contact 3-manifold is diffeomorphic to $S^3_{-n-1}(K_{-2p})$. 
EMBEDDED VERSION:
An **open book decomposition** of $M$ is a pair $(B, \pi)$ where

1. $B$ is an oriented link in $M$ called the **binding** of the open book, and

2. $\pi : M \setminus B \longrightarrow S^1$ is a fibration of the complement of $B$ such that
   (i) $\pi^{-1}(\theta)$ is the interior of a compact surface $\Sigma_\theta \subset M$ and
   (ii) $\partial \Sigma_\theta = B$ for all $\theta \in S^1$.

The surface $\Sigma = \Sigma_\theta$, for any $\theta$, is called the **page** of the open book.
ABSTRACT VERSION:
Suppose $\Sigma$ is a compact bordered surface, and $f$ is an orientation preserving automorphism of $\Sigma$ fixing $\partial \Sigma$.

Let $M_f$ be $\Sigma \times [0,1]/\sim$, where $(x,1) \sim (f(x),0)$ for $x \in \Sigma$, and $(y,t) \sim (y,t')$ for $y \in \partial \Sigma$, $t, t' \in [0,1]$.

If $M_f$ is diffeomorphic to $M$, then we call the pair $(\Sigma, f)$ is an abstract open book decomposition of $M$, and call $f$ the monodromy.
Examples of open book decompositions of $S^3$
Definition (Thurston-Winkelnkemper)

A contact structure $\xi$ on $M$ is **supported** by an embedded open book decomposition $(B, \pi)$ of $M$ if there is a contact 1-form $\alpha$ for $\xi$ such that

1. $d\alpha$ is a positive area form on each page $\Sigma_\theta$ of the open book and
2. $\alpha > 0$ on $B$ (Recall: $B$ and the pages are oriented).
Definition (Thurston-Winkelnkemper)

A contact structure \( \xi \) on \( M \) is \textbf{supported} by an embedded open book decomposition \((B, \pi)\) of \( M \) if there is a contact 1-form \( \alpha \) for \( \xi \) such that

1. \( d\alpha \) is a positive area form on each page \( \Sigma_\theta \) of the open book and
2. \( \alpha > 0 \) on \( B \) (Recall: \( B \) and the pages are oriented).

Every embedded open book decomposition can be transformed to an abstract one. If \((B, \pi)\) can be transformed to an abstract open book decomposition \((\Sigma, f)\), then we say that \( \xi \) is \textbf{supported} by \((\Sigma, f)\).
Main result II

\( \Sigma \): a cpct planar surface with \( n + p + q + 1 \) bdry cpnts.

\( c_0, c_1, \ldots, c_{n+p+q} \): components of \( \partial \Sigma \).

\( \Phi : = \tau_{m_1}^{m_1} \tau_{m_2}^{m_2} \cdots \tau_{n+q-1}^{m_{n+q-1}} \tau_{n+q+1}^{m_{n+q+1}} \cdots \tau_{n+p+q}^{m_{n+p+q}} \tau_{B_1} \tau_{B_2} \).

\( \tau_i \): positive Dehn twist along the bdry cpnt \( c_i \).
Main result II

Theorem (Kaloti-L.)

Let \((M, \xi)\) be the contact 3-manifold supported by the open book decomposition \((\Sigma, \Phi)\). Then the contact 3-manifold \((M, \xi)\) admits a unique Stein filling up to diffeomorphism.
Corollary (Kaloti-L.)

Let $(M, \xi)$ be the contact 3-manifold obtained by Legendrian surgery on the following Legendrian link. Then the contact 3-manifold $(M, \xi)$ admits a unique Stein filling up to diffeomorphism.
Tools used in the proof i: Open book decom., Lefschetz fibration and Stein filling

Theorem (Loi-Piergallini, Akbulut-Ozbagci, Giroux)

Stein domain $\leftrightarrow$ Positive allowable Lefschetz fibration

boundary $\leftrightarrow$ boundary

Contact structure $\leftrightarrow$ Open book decomposition

Youlin Li (With Amey Kaloti)
Theorem (Wendl)

Suppose \((M, \xi)\) is supported by a \textit{planar} open book decomposition. Then every Stein filling of \((M, \xi)\) is symplectic deformation equivalent to a positive allowable Lefschetz fibration compatible with the given open book decomposition of \((M, \xi)\).
1, Abelianization of mapping class groups for planar compact surfaces,
1, Abelianization of mapping class groups for planar compact surfaces,

2, Stretch factors of pseudo-Anosov diffeomorphisms.
Thank you!