ON THE THURSTON-BENNEQUIN NUMBER OF KNOTS AND LINKS IN LEGENDRIAN GRAPHS

Toshifumi Tanaka
Gifu University

August 26, 2014
Outline:

- A contact structure
- Legendrian knots
- The Thurston-Bennequin number
- Legendrian graphs
- An invariant of Legendrian graphs
- Recent works
- Results
A contact structure

- Let $M$ be an oriented 3-manifold and $\xi$, a 2-plane field on $M$.

We say $\xi$ is a contact structure on $M$ if $\xi = \ker \alpha$ for some 1-form $\alpha$ satisfying $\alpha \wedge d\alpha > 0$:

Example: On $\mathbb{R}^3$, the 1-form $= dz - ydx$ gives the standard contact structure on $\mathbb{R}^3$, $\xi_{\text{std}}$. 
The standard contact structure

\[ \xi_{\text{std}} = \text{Ker}(dz - ydx) \]
A standard contact structure
A standard contact structure
A standard contact structure
A standard contact structure
A standard contact structure
A standard contact structure
A standard contact structure
Legendrian knots

• A knot in $\mathbb{R}^3$, is a simple closed curve
  $\gamma : S^1 \to \mathbb{R}^3$

• A knot $\gamma$ in $(\mathbb{R}^3, \xi_{\text{std}})$ is called Legendrian if for all $p \in \gamma$ and $\xi_p$ the contact 2-plane at $p$, $T_p \gamma \subset \xi_p$. 
Legendrian isotopy

- A **Legendrian isotopy** between Legendrian knots is an ambient isotopy with each level Legendrian.

We study Legendrian knots up to Legendrian isotopy.
Front projection

• The \((x,z)\)-projection of a Legendrian knot is called a **front projection**.

Front projections of Legendrian knots do not have vertical tangencies (since \(y = \frac{dz}{dx}\)). At each crossing the overstrand is always the one with smaller slope (since the \(y\)-axis points away from the viewer).
Front projection
Front projection
A classical invariant of Legendrian knots

The Thurston-Bennequin number: $tb$
The Thurston-Bennequin number

• Let $K$ be an oriented Legendrian knot in $\mathbb{R}^3$.
• In a front projection, we define the Thurston-Bennequin number $tb(K)$ by
  
  \[ tb(K) = \text{writhe} - \frac{1}{2} \#\text{cusps} \]

  \text{writhe} = \text{signed count of crossings in the projection}

• The Thurston-Bennequin number measures the amount of twisting of the contact planes along the knot and does not depend on the chosen orientation of $K$. 
Example.

\[ tb(K_1) = -1, \quad tb(K_2) = -2, \quad tb(K_3) = -2 \]
The maximal Thurston-Bennequin number

The maximal Thurston-Bennequin number of a knot $L$ is the maximal value, denoted by $TB(L)$, of the Thurston-Bennequin numbers for Legendrian knots which are topologically isotopic to $L$.

Theorem. (Fuchs-Tabachnikov, Topology 36 (1997)). Let $L$ be a knot in $\mathbb{R}^3$. Then $\quad TB(L) < -\max_{x} \deg_{y} F(x,y) (L)$. 
Example.

$xz$-plane

$TB(L) = 1$, $TB(O) = -1$. 

$L$
Legendrian graphs

• A Legendrian graph in \((\mathbb{R}^3, \xi_{\text{std}})\) is a graph embedded in such a way that all its edges are Legendrian segments.
• Such an embedding is called a Legendrian embedding.
An invariants of Legendrian graphs

We extend $tb$ to Legendrian graphs.

We can define the invariant for cycles in a Legendrain graph (which is piecewise smooth Legendrian knots) in appropriate way (O-P).

- For a Legendrian graph $\Gamma$:
  
  $tb(\Gamma) = \text{the set of the } tbs \text{ of the cycles of } \Gamma$. 
Legendrian graphs

\[ \text{tb}(L) = \{-1, -1, -1, -2\}. \]
Recent works

Theorem (O'Donnol-Pavelescu, A. G. T. 12 (2012)). Any spatial graph has a Legendrian embedding in $(\mathbb{R}^3, \xi_{\text{std}})$. 
Recent works

Theorem (O'Donnol-Pavelescu, A. G. T. 12 (2012)). A graph $G$ admits a Legendrian embedding in $(\mathbb{R}^3, \xi_{\text{std}})$ with all its cycles trivial unknots if and only if $G$ does not contain $K_4$ as a minor.

A graph $H$ is a minor of a graph $G$ if $H$ can be obtained from $G$ by a finite number of edge contractions.

A trivial unknot $= \includegraphics[width=0.3\textwidth]{trivial_unknot}$

$TB(O) = -1.$
Recent works

Theorem (O'Donnol-Pavelescu, A. G. T. 12 (2012)). Let $G$ be a graph that contains $K_4$ as a minor. There does not exist a Legendrian embedding of $G$ such that all its cycles realize the maximal Thurston-Bennequin numbers that only consists of odd numbers.
Problem

Does there exist a Legendrian embedding of $K_4$ such that all its cycles are knots realizing their maximal Thurston-Bennequin numbers?
Results

Theorem.
There exists an infinitely many Legendrian embeddings of $K_4$ such that all its cycles realize the maximal Thurston-Bennequin numbers that consist of six odd numbers and one even number.
Results

There exists a Legendrian embedding $L$ of $K_4$ such that
a) $tb(L) = \{-1, -1, -1, 0, 1, 5, 5\}$,
b) Every cycle realizes the maximal Thurston-Bennequin number.
Results

There exist Legendrian embeddings $L_n$ of $K_4$ such that

a) $tb(L_n) = \{-1, -1+2n, -1+2n, 0, 1+2n, 5+2n, 5\}$ ($n \in \mathbb{Z}$, $n > 0$),

b) every cycle realizes the maximal Thurston-Bennequin number.
An invariant of spatial graphs

- We say that a spatial graph is \textit{mTB-realizable} if it is ambient isotopic to a Legendrian graph such that all its cycles realize their mTB’s.
An invariant of spatial graphs

- We say that a spatial graph is mTB-realizable if it is ambient isotopic to a Legendrian graph such that all its cycles realize their mTB’s.

(a) is not mTB-realizable (by a result of O'Donnol and Pavelescu.)
Corollary. There exists an infinite family of Legendrian embedding of $K_4$ such that they are mTB-realizable.
Results

Proposition.
If a finite graph $G$ contains two cycles that have no common edges and no common vertices then there exists an embedding of $G$ such that it is not mTB-realizable.
Results

Proposition.
If a finite graph $G$ contains two cycles that have no common edges and no common vertices then there exists an embedding of $G$ such that it is not mTB-realizable.

Thank you very much