Local Moves on Knots and Polynomial Invariants

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Knots and Low Dimensional Manifolds
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There are several local moves on knots and links such as a crossing change, a band surgery, and an $H(2)$-move. We focus on a crossing change.

The Gordian distance of two knots is the minimum number of crossing changes needed to transform one into the other. We introduce some methods giving a lower bound of the Gordian distance using the Jones polynomial.

This is partially a joint work with Hiromasa Moriuchi.
Crossing change and unknotting number

By changing some of the crossings from over to under or vice versa, any projection of a knot can be made into a projection of the unknot.

![Diagram of crossing change](image)

The **unknotting number** of a knot $K$, $u(K)$, is the minimum number of crossing changes that are necessary to transform $K$ into the trivial knot, where the minimum taken over all projections.

**Example 1**

The trefoil knot has unknotting number one, $u(3_1) = 1$. 

![Diagram of trefoil knot](image)
Gordian distance

The **Gordian distance** from a knot $K$ to another knot $K'$, $d(K, K')$, is the minimum number of crossing changes that are necessary to transform $K$ into $K'$.

**Example 2**

d$(5_1, 3_1) = 1$. 

![Diagram](image_url)
### Table of Gordian distances

<table>
<thead>
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<th>0₁</th>
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Darcy’s Gordian distance table

- (Torisu, Darcy-Sumners, Berge) Necessary and sufficient condition for 2-bridge knots $K, K'$ with $d(K, K') = 1$.

Upper bounds.
- Triangle inequality: $d(K, K') \leq u(K) + u(K')$, etc.
- Inspection of a knot diagram.

Lower bounds.
- Signature.
- (H. Murakami) Necessary condition for a knot $K$ with $d(K, \text{a 2-bridge knot}) = 1$ using a quadratic form.
Skein triple

Let $L_+, L_-, L_0$ be three links that are identical except near one point where they are as in

We call $(L_+, L_-, L_0)$ a **skein triple**.
Skein triple and local moves

Let \((L_+, L_-, L_0)\) be a skein triple.

Then \(L_+\) and \(L_-\) are related by a crossing change;
\(L_+\) and \(L_0\), \(L_-\) and \(L_0\) are related by a coherent band surgery.

\[ \text{Crossing change} \]

\[ \text{Coherent band surgery} \]
Let $L$ be an oriented link. Let $M$ a Seifert matrix of $L$, and $A = M + M^T$. Then the signature of $L$, $\sigma(L)$, is defined by:

$$\sigma(L) = \#(\text{positive eigenvalues of } A) - \#(\text{negative eigenvalues of } A).$$

- $\sigma(\text{the unknot}) = \sigma(\text{a trivial link}) = 0$.
- $\sigma(3_1; \text{the negative trefoil knot}) = +2$.
- $\sigma(\text{a knot})$ is an even integer.

Properties of signature

Proposition 1

(i) If two links $L$ and $L'$ are related by a coherent band surgery, then

$$|\sigma(L) - \sigma(L')| \leq 1.$$  \hfill (1)

(ii) For oriented links $L_1$ and $L_2$, we have

$$\sigma(L_1 \# L_2) = \sigma(L_1) + \sigma(L_2).$$  \hfill (2)

Proposition 2

Let $(L_+, L_-, L_0)$ be a skein triple. Then

$$\sigma(L_\pm) - \sigma(L_0) \in \{-1, 0, 1\}; \hfill (3)$$
$$\sigma(L_-) - \sigma(L_+) \in \{0, 1, 2\}. \hfill (4)$$
Methods for giving a lower bound for the Gordian distance: Special values of the polynomial invariants. (1/2)

- The Jones polynomial.
  - Miyazawa’s criterion.
  - Traczyck’s method (Proposition 4).
  - New criteria (Theorems 5 and 6).

- HOMFLYPT polynomial.
  - Miyazawa’s criterion
  - New criterion.
Methods for giving a lower bound for the Gordian distance: Special values of the polynomial invariants. (2/2)

◊ Q polynomial.
  ● Stoimenow’s criterion

  ● New criterion.
Other methods for giving a lower bound for the Gordian distance

◊ Stoimenow’s criterion using the linking form.  

◊ Nakanishi and Okada’s criterion.  
Jones polynomial $V(L) = V(L; t) \in \mathbb{Z}[t^{\pm 1/2}]$

An invariant of the isotopy type of an oriented link $L$ defined by:

- $V(U) = 1$;
- $t^{-1}V(L_+) - tV(L_-) = \left(t^{1/2} - t^{-1/2}\right)V(L_0)$,

where $U$ is the unknot and $(L_+, L_-, L_0)$ is a skein triple:
Value of the Jones polynomial at $t = e^{i\pi/3} (= \omega)$

For a $c$-component link $L$

$$V(L; e^{i\pi/3}) = \pm i^{c-1}(i\sqrt{3})^d,$$

where

- $d = \dim H_1(\Sigma(L); \mathbb{Z}_3)$ with $\Sigma(L)$ the double cover of $S^3$ branched over $L$;
- $V(L; e^{i\pi/3})$ means $V(L; t)$ at $t^{1/2} = e^{i\pi/6}$.

Value of the Jones polynomial at $t = e^{i\pi/3} = \omega$

Let $(L_+, L_-, L_0)$ be a skein triple.

<table>
<thead>
<tr>
<th>Cases</th>
<th>$V(L_+; \omega)/V(L_-; \omega)$</th>
<th>$V(L_0; \omega)/V(L_-; \omega)$</th>
<th>$V(L_0; \omega)/V(L_+; \omega)$</th>
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<tr>
<td>(a)</td>
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<td>(b)</td>
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Proposition 3

If two knots $K$ and $K'$ are related by a crossing change, then

$$V(K; \omega)/V(K'; \omega) \in \left\{ \pm 1, \pm i\sqrt{3}^{\pm 1} \right\}.$$ 

Thus, if $|V(K; \omega)/V(K'; \omega)| = \sqrt{3}^e$, then $d(K, K') \geq |e|$.

Example: $u(7_4) = 2$


Proof of $u(7_4) = 2$ by Traczyk

Let $K = 7_4$. Suppose $u(K) = 1$. Then since $\sigma(K) = -2$, $K$ is unknotted by changing a positive crossing, that is, $\exists$ a skein triple $(K, U, L_0)$ for some link $L_0$.

Since $V(K; \omega) = -i\sqrt{3}$, $V(U; \omega) = 1$, from the table below there is not such a skein triple.

<table>
<thead>
<tr>
<th>Cases</th>
<th>$V(L_+; \omega)/V(L_-; \omega)$</th>
<th>$V(L_0; \omega)/V(L_-; \omega)$</th>
<th>$V(L_0; \omega)/V(L_+; \omega)$</th>
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<td>$i\sqrt{3}^{-1}$</td>
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A generalization of Traczyk’s example

Miyazawa generalized Traczyk’s example.

**Proposition 4**

If two knots $K$ and $K'$ satisfy $\sigma(K) - \sigma(K') = -2n$ and $V(K; \omega)/V(K'; \omega) = -(i \sqrt{3}^{\pm 1})^n$, where $n$ is a positive integer, then $d(K, K') > n$.

Criterion for knots with distance one

Theorem 5

Suppose that two knots $K$ and $K'$ are related by a crossing change. If $V(K; \omega) = \eta V(K'; \omega) = \pm(i\sqrt{3})^\delta$, $\eta = \pm 1$, then $V(K; -1) \equiv \eta V(K'; -1) \pmod{3^{\delta+1}}$.

Example 3

d(8_5, 3_1!) > 1.

$\sigma(3_1!) = -2$, $\sigma(8_5) = -4$;

$V(8_5; \omega) = V(3_1!; \omega) = i\sqrt{3}$;

$V(8_5; -1) = 21 \not\equiv V(3_1!; -1) = -3 \pmod{3^2}$.

T. Kanenobu and H. Moriuchi,

*Links which are related by a band surgery or crossing change,*

Example. \( d(8_5, 3_1!) = 2 \)

We have \( d(8_5, 3_1!) \leq 2. \)

\( d(8_5, 3_1!\#3_1!) = 1; \)

Then \( d(8_5, 3_1!) \leq d(8_5, 3_1!\#3_1!) + d(3_1!\#3_1!, 3_1!) = 2. \)
Proof of $d(3_1!, 8_5) > 1$

Let $K' = 3_1!, K = 8_5$. Suppose $d(K, K') = 1$. Then since $\sigma(3_1!) = -2$, $\sigma(8_5) = -4$, there exists a 2-component link $L$ such that $(K, K', L)$ is a skein triple. $V(K; \omega) = V(K'; \omega) = i\sqrt{3}$.

<table>
<thead>
<tr>
<th>Cases</th>
<th>$V(K; \omega)/V(K'; \omega)$</th>
<th>$V(L; \omega)/V(K'; \omega)$</th>
<th>$V(L; \omega)/V(K; \omega)$</th>
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$\implies V(L; \omega) = (-\sqrt{3})(i\sqrt{3}) = -3i = i(i\sqrt{3})^2$. $\implies \dim H_1(\Sigma(L); \mathbb{Z}_3) = 2$, where $\Sigma(L)$ the double cover of $S^3$ branched over $L$. $\implies \det(L) \equiv 0 \pmod{3^2}$. Putting $t = -1$ in $t^{-1}V(K) - tV(K') = (t^{1/2} - t^{-1/2}) V(L)$, we have: $-V(K; -1) + V(K'; -1) = 2iV(L; -1)$. Then since $V(K; -1) = 21$, $V(K'; -1) = -3$, we have $\det(L) = |V(L; -1)| = 12 \not\equiv 0 \pmod{9}$, a contradiction.
Criterion for knots with Gordian distance two

Theorem 6
Let $K$ and $K'$ be oriented knots with $d(K, K') = 2$ and $\sigma(K) - \sigma(K') = 4$. Then if $V(K; \omega) = V(K'; \omega) = \pm (i \sqrt{3})^\delta$, then $V(K; -1) \equiv V(K'; -1) \pmod{3^{\delta+1}}$.

Example 4
$d(3_1, 7_4) = 3$.

$d(3_1, 7_4) > 2$.

In fact, $\sigma(3_1) = 2$, $\sigma(7_4) = -2$;

$V(3_1; \omega) = V(7_4; \omega) = -i \sqrt{3}$;

$V(3_1; -1) = -3 \not\equiv V(7_4; -1) = -15 \pmod{3^2}$.

$d(3_1, 7_4) \leq u(3_1) + u(7_4) = 1 + 2 = 3$. 
Proof of $d(3_1, 7_4) > 2$.

Let $K = 3_1$, $K' = 7_4$. Suppose $d(K, K') = 2$. Since $\sigma(K) = 2$, $\sigma(K') = -2$, there exists a knot $J$ such that $J$ is obtained from $K'$ by changing a positive crossing, and $K$ is obtained from $J$ by changing a positive crossing.

$$V(K; \omega) = V(K'; \omega) = -i\sqrt{3}$$

$$\implies V(J; \omega) = \eta V(K; \omega), \; \eta = \pm 1.$$  

<table>
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<tr>
<th>Cases</th>
<th>$V(L_+; \omega)/V(L_-; \omega)$</th>
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By Theorem 5, $V(K; -1) \equiv \eta V(J; -1) \pmod{3^2}$ and $V(K'; -1) \equiv \eta V(J; -1) \pmod{3^2}$. 

$$\implies V(K; -1) = -3 \equiv V(K'; -1) = -15 \pmod{3^2},$$
a contradiction.
\[ L = 9_3^2 = L9a30, \ u(L) = 3 \]

Kohn has given a table of unlinking number for 2 component prime links with up to 9 crossing. There remain 5 links whose unlinking number have not been settled, and \( L \) is one of them.

- \(2 \leq u(L) \leq 3\)
- \(\sigma(L) = 3\)
- Conway polynomial
  \[ \nabla(L) = -z + 4z^3 + 2z^5 \]
- Linking number \( \text{lk}(L) = -1 \)

Unlinking number and Splitting number

- The **unlinking number** $u(L)$ of a link $L$ is the minimal number of crossing changes necessary to convert $L$ into a trivial link, where this minimum is taken over all diagrams of $L$.

- The **splitting number** of a link is the minimal number of crossing changes required, on any diagram, to convert it to a split link.

i) Crossing changes of a component with itself are permitted.

ii) Crossing changes are only between different components.
Criterion for links with Gordian distance two

Theorem 7

Let $L$ and $L'$ be oriented $c$-component links with $d(L, L') = 2$ and $\sigma(L) - \sigma(L') \geq 3$. Then if

$V(L; \omega) = V(L'; \omega) = \pm i^{c-1}(i\sqrt{3})^\delta$, then

$i^{c-1}V(L; -1) \equiv i^{c-1}V(L'; -1) \pmod{3^{\delta+1}}$.

Example 5

$L = 9_3^2 = L9a30$. $u(L) = d(L, U^2) > 2$, $U^2$ the trivial 2-component link.

In fact,

$\sigma(L) = 3, \sigma(U^2) = 0$;

$V(L; \omega) = V(U^2; \omega) = -\sqrt{3}$;

$iV(L; -1) = -30 \not\equiv iV(U^2; -1) = 0 \pmod{3^2}$.
## Improved table of Gordian distances (1/2)

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Improved table of Gordian distances (2/2)

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Proof of Eq. (4)

Since $L_-$ is obtained from $L_+$ by changing a positive crossing, $L_+$ yields the product link $L_- \# H_+$, $H_+$ is a positive Hopf link, by doing a coherent band surgery as shown in:

Then by Eq. (1) we have

$$|\sigma(L_+) - \sigma(L_- \# H_+)| \leq 1.$$  \hspace{1cm} (5)

Since

$$\sigma(L_- \# H_+) = \sigma(L_-) + \sigma(H_+) = \sigma(L_-) - 1$$  \hspace{1cm} (6)

by Eq. (2), we obtain Eq. (4).