A construction of a knot projection satisfying the equation from Euler Characteristic

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joint work with Kokoro Tanaka (Tokyo Gakugei University)
Preliminary

$G$ : connected loopless planar graph on $S^2$

- $f_n(G) := \# \text{ of } n\text{-sided faces (}n\text{-gons) of } G$
$G$ : connected loopless planar graph on $\mathbb{S}^2$

- $f_n(G) := \#$ of $n$-sided faces ($n$-gons) of $G$

Example

![Graph](image_url)
G : connected loopless planar graph on $S^2$

- $f_n(G) := \#$ of $n$-sided faces ($n$-gons) of $G$

Example

A knot projection satisfying the equation from Euler formula
$G$: connected loopless planar graph on $S^2$

- $f_n(G) := \#$ of $n$-sided faces ($n$-gons) of $G$

Example

$$G$$

![Diagram of a graph with a 3-gon highlighted]
\( G \): connected loopless planar graph on \( S^2 \)

- \( f_n(G) := \# \) of \( n \)-sided faces (\( n \)-gons) of \( G \)

**Example**

\[
f_n(G) = \begin{cases} 
2 & (n = 2) \\
5 & (n = 3) \\
1 & (n = 4) \\
1 & (n = 5) \\
0 & (\text{otherwise}) 
\end{cases}
\]
Assumption

Projections and diagrams are connected and reduced.

\[ P : \text{knot/link projection} \quad D : \text{knot/link diagram} \]
Assumption

Projections and diagrams are connected and reduced.

\[ P : \text{knot/link projection} \]
\[ D : \text{knot/link diagram} \]
\[ G : 4\text{-regular planar graph} \]
Assumption

Projections and diagrams are connected and reduced.

$P : \text{knot/link projection}$

$D : \text{knot/link diagram}$

\[ f_n(P) : \text{the number of } n\text{-gons of } P \]
\[ f_n(D) : \text{the number of } n\text{-gons of } D \]

\[ f_n(P) = f_n(D) = \begin{cases} 
2 & (n = 2) \\
5 & (n = 3) \\
1 & (n = 4) \\
1 & (n = 5) \\
0 & (\text{otherwise}) 
\end{cases} \]
Definition

\( D \) : diagram/projection of a knot/link

\( D \) is an \((a_1, a_2, a_3, \ldots)\)-diagram/projection

\( \iff \) each face of \( D \) is an \(a_n\)-gon for some \(a_n\) that appears in the sequence \((a_1, a_2, a_3, \ldots)\).
\( D \): diagram/projection of a knot/link

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\( \iff \) each face of \( D \) is an \( a_n \)-gon for some \( a_n \) that appears in the sequence \((a_1, a_2, a_3, \ldots)\).
Implication from Euler’s formula

$G$ : connected loopless 4-regular planar graph on $\mathbb{S}^2$

By Euler’s formula $v - e + f = 2$, we obtain the following equation:

$$2f_2(G) + f_3(G) = 8 + f_5(G) + 2f_6(G) + 3f_7(G) + \cdots$$

$$\Leftrightarrow \sum_{i=2}^{\infty} (4 - n)f_n(G) = 8$$
Implication from Euler’s formula

$G$ : connected loopless 4-regular planar graph on $\mathbb{S}^2$

By Euler’s formula $v - e + f = 2$, we obtain the following equation:

$$2f_2(G) + f_3(G) = 8 + f_5(G) + 2f_6(G) + 3f_7(G) + \cdots$$

$$\iff \sum_{i=2}^{\infty} (4 - n)f_n(G) = 8$$

Easy consequence

$G$ : connected loopless planar 4-regular graph on $\mathbb{S}^2$

- $f_2(G) \neq 0$ or $f_3(G) \neq 0$
- $\sum_{n \in \mathbb{N}} f_{2n+1}(G)$ is even
$D$: diagram/projection of a knot/link on $S^2$

The same equation holds for $D$:  

$$2f_2(D) + f_3(D) = 8 + f_5(D) + 2f_6(D) + 3f_7(D) + \cdots$$  

$$\iff \sum_{i=2}^{\infty} (4 - n)f_n(D) = 8$$

**Easy consequence**

$D$: diagram/projection of a knot/link on $S^2$

- $f_2(D) \neq 0$ or $f_3(D) \neq 0$
- $\sum_{n \in \mathbb{N}} f_{2n+1}(D)$ is even
Typically it is assumed that graphs are simple.

\[ f_3 = 8 + f_5 + 2f_6 + 3f_7 + (\text{multi-edges}) \]
Known results on 4-regular graph on $S^2$

Typically it is assumed that graphs are simple. In graph theory, there is a long history of investigations into the sequences of values that satisfy the equation

$$f_3 = 8 + f_5 + 2f_6 + 3f_7 + \cdots (\ast)$$

and that represent connected 4-regular simple planar graphs on $S^2$.

**Remark**

$f_4$ (number of 4-gons) is not restricted by the equation from Euler formula.
Assume that graphs are connected and simple.

**Theorem (Grüenbaum 1969)**

\[ \forall \{f_3, f_5, f_6, \ldots \} : \text{a seq. of non negative integers satisfying (*)}, \]
\[ \exists f_4 : \text{non negative integer} \]
\[ \exists G : 4\text{-regular graph on } S^2 \text{ s.t. } f_n(G) = f_n \]

\[ f_3 = 8 + f_5 + 2f_6 + 3f_7 + \cdots (\ast) \]
Assume that graphs are connected and simple.

**Theorem (Grünbaum 1969)**

\( \forall \{f_3, f_5, f_6, \ldots \} : \) a seq. of non negative integers satisfying \((\ast)\),

\( \exists f_4 : \) non negative integer

\( \exists G : 4\text{-regular graph on } \mathbb{S}^2 \text{ s.t. } f_n(G) = f_n \)

\[ f_3 = 8 + f_5 + 2f_6 + 3f_7 + \cdots (\ast) \]

**Theorem (Jeong 1995)**

\( \forall \{f_3, f_5, f_6, \ldots \} : \) a seq. of non negative integers satisfying \((\ast)\),

\( \exists f_4 : \) non negative integer

\( \exists G : \text{cut-through Eulerian 4\text{-regular graph on } } \mathbb{S}^2 \text{ s.t. } f_n(G) = f_n \)
Assume that graphs are connected and simple.

**Theorem (Grünbaum 1969)**

$\forall \{f_3, f_5, f_6, \cdots \}$: a seq. of non negative integers satisfying $(\ast)$,

$\exists f_4$ : non negative integer

$\exists G$ : 4-regular graph on $S^2$ s.t. $f_n(G) = f_n$

\[ f_3 = 8 + f_5 + 2f_6 + 3f_7 + \cdots (\ast) \]

**Theorem (Jeong 1995)**

$\forall \{f_3, f_5, f_6, \cdots \}$: a seq. of non negative integers satisfying $(\ast)$,

$\exists f_4$ : non negative integer

$\exists G$ : knot projection on $S^2$ s.t. \[ \begin{cases} f_n(G) = f_n & (n \neq 2) \\ f_2(G) = 0 \end{cases} \]
Assume that graphs are connected and **simple**.

**Theorem (Grünbaum 1969)**

\[
\forall \{f_3, f_5, f_6, \ldots \} : \text{a seq. of non negative integers satisfying (\(*\)),} \\
\exists f_4 : \text{non negative integer} \\
\exists G : 4\text{-regular graph on } \mathbb{S}^2 \text{ s.t. } f_n(G) = f_n
\]

\[
f_3 = 8 + f_5 + 2f_6 + 3f_7 + \cdots (\ast)
\]

**Theorem (Jeong 1995)**

\[
\forall \{f_3, f_5, f_6, \ldots \} : \text{a seq. of non negative integers satisfying (\(*\)),} \\
\exists f_4 : \text{non negative integer} \\
\exists G : \text{knot projection on } \mathbb{S}^2 \text{ s.t. } \begin{cases} f_n(G) = f_n & (n \neq 2) \\ f_2(G) = 0 \end{cases}
\]
A purposes of our study

Question A

∀K : knot/link
∀{f_2, f_3, f_5, f_6 \cdots} : seq. of non negative integers satisfying (**)

∃?f_4 : non negative integer
∃?P : projection of K s.t. f_n(P) = f_n

(**) 2f_2 + f_3 = 8 + f_5 + 2f_6 + 3f_7 + \cdots

Recall [Theorem (Jeong 1995)]

∀{f_3, f_5, f_6 \cdots} : seq. of non negative integers satisfying (*),

∃f_4 : non negative integer
∃G : knot projection on S^2 s.t.
\[ \begin{cases} f_n(G) = f_n & (n \neq 2) \\ f_2(G) = 0 \end{cases} \]

(*) f_3 = 8 + f_5 + 2f_6 + 3f_7 + \cdots
Known results on knot/link projections on $\mathbb{S}^2$

<table>
<thead>
<tr>
<th>Theorem (Adams-S-Tanaka 2011)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Every knot/link has a $(3, 4, n)$-projection $(\forall n \geq 5)$.</td>
</tr>
<tr>
<td>2. Every knot/link has a $(2, 4, 5)$-projection.</td>
</tr>
</tbody>
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<th>Theorem (S)</th>
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</tr>
</tbody>
</table>
Main result

Main Theorem (S-Tanaka)

\[ \forall P: \text{ knot projection with } \]
\[ \forall K: \text{ knot/link} \]
\[ \exists P': \text{ projection of } K \text{ s.t. } f_n(P') = f_n(P) \quad (n \neq 4) \]

\[ f_n(P) = \begin{cases} 
8 & (n = 3) \\
6 & (n = 4) \\
0 & (\text{otherwise}) 
\end{cases} \]
Since there exists a knot projection $P$ with

\[
\begin{align*}
\text{s.t. } f_n(P) &= \begin{cases} 
8 & (n = 3) \\
6 & (n = 4) \\
0 & \text{(otherwise)}
\end{cases}
\end{align*}
\]

we obtain the following:

Recall [Theorem (S)]

Every knot/link has a (3, 4)-projection.

as a corollary of our main theorem.

Recall [Main Theorem (S-Tanaka)]

\[
\begin{align*}
\forall P : \text{ knot projection with } & \quad \begin{array}{c}
\begin{array}{c}
0 \\
1 \\
2
\end{array}
\end{array} \\
\forall K : \text{ knot/link} \\
\exists P' : \text{ projection of } K \text{ s.t. } f_n(P') = f_n(P) & \quad (n \neq 4)
\end{align*}
\]
Since $\exists (2, 5)$-projection $P$ of a knot with

By our main theorem we obtain the following:

Recall [Theorem(Adams-S-Tanaka 2011)]

$\forall$ knot/link has a $(2,4,5)$-projection
The knot projection $P$ in Jeong’s theorem has $P$. 

Recall [Theorem (Jeong 1995)]

$\forall \{f_3, f_5, f_6 \cdots \} : \text{a seq. of non negative integers satisfying (\text{*}),}$

$\exists f_4 : \text{non negative integer}$

$\exists P : \text{knot projection on } \mathbb{S}^2 \text{ s.t. } \begin{cases} f_n(P) = f_n \quad (n \neq 2) \\ f_2(P) = 0 \end{cases}$
The knot projection $P$ in Jeong’s theorem has

\[ f_3(P) = f_5(P) = f_6(P) = \cdots \]

Recall [Theorem (Jeong 1995)]
\[
\forall \{f_3, f_5, f_6, \ldots \} : \text{a seq. of non negative integers satisfying } (\ast), \quad \exists f_4 : \text{non negative integer}
\exists P : \text{knot projection on } S^2 \text{ s.t. } \begin{cases} f_n(P) = f_n \quad (n \neq 2) \\ f_2(P) = 0 \end{cases}
\]

Recall [Main Theorem (S-Tanaka)]
\begin{align*}
\forall P : \text{knot projection with} \\
\forall K : \text{knot/link} \\
\exists P' : \text{projection of } K \text{ s.t. } f_n(P') = f_n(P) \quad (n \neq 4)
\end{align*}
Corollary

\( \forall K : \text{knot/link} \)
\( \forall \{f_3, f_5, f_6, \ldots\} : \text{a seq. of non negative integers satisfying (\(\ast\))}, \)
\( \exists f_4 : \text{non negative integer} \)
\( \exists D : \text{diagram of } K \text{ s.t. } f_n(D) = \begin{cases} 
0 & (n = 2) \\
 f_n & (n \neq 2) 
\end{cases}, \)

\((\ast) \quad f_3 = 8 + f_5 + 2f_6 + 3f_7 + \cdots \)
Corollary

∀K : knot/link
∀\{f_3, f_5, f_6 \cdots \} : a seq. of non negative integers satisfying (\star),
∃f_4 : non negative integer
∃D : diagram of K s.t. \ f_n(D) = \begin{cases} 
0 & (n = 2) \\
f_n & (n \neq 2), 
\end{cases}

(\star) \ f_3 = 8 + f_5 + 2f_6 + 3f_7 + \cdots
Recall [Main Theorem (S-Tanaka)]

\[ \forall P : \text{knot projection with} \]
\[ \forall K : \text{knot/link} \]
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Recall [Main Theorem (S-Tanaka)]

∀P : knot projection with

∀K : knot/link

∃D : diagram of K s.t. \( f_n(D) = f_n(P) \) \( (n \neq 4) \)

1. We give a quasi-toric braid representation of K.
Theorem (Manturov 2002)

\( \forall \) knot/link has a quasi-toric braid representation.
Recall[Main Theorem (S-Tanaka)]

∀\(P\): knot projection with
\[1, 2\]
∀\(K\): knot/link

∃\(D\): diagram of \(K\) s.t. \(f_n(D) = f_n(P)\) \((n \neq 4)\)

1. We give a quasi-toric braid representation of \(K\).
Recall [Main Theorem (S-Tanaka)]

\[ \forall P: \text{knot projection with} \]

\[ \forall K: \text{knot/link} \]

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1. We give a quasi-toric braid representation of \( K \).
2. From \( P \), we construct a knot projection \( P_0 \) with

\[ \begin{array}{cccccc}
\text{0} & \text{1} & \text{2} & \cdots & \text{N-1} & \text{N} \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\
\text{s.t. } f_n(P_0) = f_n(P) \quad (n \neq 4) \\
\end{array} \]
Recall [Main Theorem (S-Tanaka)]

∀P: knot projection with

∀K: knot/link

∃D: diagram of K s.t. \( f_n(D) = f_n(P) \) \( (n \neq 4) \)

1. We give a quasi-toric braid representation of K.
2. From \( P \), we construct a knot projection \( P_0 \) with

3. From \( P_0 \), we construct a diagram of unknot with writhe 0.
Recall [Main Theorem (S-Tanaka)]

∀P : knot projection with

∀K : knot/link

∃D : diagram of K s.t. \( f_n(D) = f_n(P) \) (\( n \neq 4 \))

1. We give a quasi-toric braid representation of \( K \).
2. From \( P \), we construct a knot projection \( P_0 \) with

   \[
   \begin{array}{cccccccc}
   & & & & & & & \\
   0 & 1 & 2 & \cdots & N-1 & N & & \\
   \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & & \\
   \end{array}
   \]

   s.t. \( f_n(P_0) = f_n(P) \) (\( n \neq 4 \))

3. From \( P_0 \), we construct a diagram of unknot with writhe 0.

From 1 and 3, we construct diagram \( D' \) of \( K \)

s.t. \( f_n(D') = f_n(P) \) (\( n \neq 3, 4, 5 \)).
Recall [Main Theorem (S-Tanaka)]

\[ \forall P : \text{knot projection with} \]
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From 1 and 3, we construct diagram \( D' \) of \( K \)

s.t. \( f_n(D') = f_n(P) \quad (n \neq 3, 4, 5) \).

By a certain trick, extra 3- and 5-gons can be canceled.
A remaining question

Question B
Can we extend the Jeong’s theorem for the sequence of non negative integers \( \{f_2, f_3, f_5, f_6 \ldots \} \) satisfying (**)?

\[
(**) \quad 2f_2 + f_3 = 8 + f_5 + 2f_6 + 3f_7 + \cdots
\]

Recall [Theorem (Jeong 1995)]

\( \forall \{f_3, f_5, f_6 \ldots \} : \) a seq. of non negative integers satisfying (*),

\( \exists f_4 : \) non negative integer

\( \exists P : \) knot projection on \( S^2 \) s.t.

\[
\begin{cases}
    f_n(P) = f_n & (n \neq 2) \\
    f_2(P) = 0
\end{cases}
\]

\[
(*) \quad f_3 = 8 + f_5 + 2f_6 + 3f_7 + \cdots
\]
Question A

∀K : knot/link,
∀\{f_2, f_3, f_5, f_6 \cdots\} : seq. of non negative integers satisfying (**)
∃? f_4 : non negative integer
∃? P : projection of K s.t. f_n(P) = f_n

(\text{**}) \quad 2f_2 + f_3 = 8 + f_5 + 2f_6 + 3f_7 + \cdots
THANK YOU!