$L$–space knots and twisting operation

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Knots and Low Dimensional Manifolds

Busan, Korea

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$M$ : rational homology 3–sphere
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$\widehat{HF}(M)$ : Heegaard Floer homology with coefficients in $\mathbb{Z}_2$ (Ozsváth-Szabó)

$$\text{rank} \widehat{HF}(M) \geq |H_1(M; \mathbb{Z})|$$
$M$: rational homology 3–sphere

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$M$ is an $L$–space if equality holds, i.e. $\text{rank} \widehat{HF}(M) = |H_1(M; \mathbb{Z})|$. 
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**Example**

Lens spaces ($\neq S^2 \times S^1$), more generally, 3–manifolds with elliptic geometry are $L$–spaces.
Dehn surgery & $L$–space

A knot $K$ is called an $L$–space knot if it admits a nontrivial Dehn surgery producing an $L$–space.
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- $K$: a nontrivial $L$–space knot

\[
K(r) \text{ is an } L\text{–space if } r \geq 2g(K) - 1 \quad \text{or} \quad r \leq -2g(K) + 1. \quad \text{(Ozsváth-Szabó)}
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A knot $K$ is called an $L$–space knot if it admits a nontrivial Dehn surgery producing an $L$–space.

- $K$: a nontrivial $L$–space knot

$K(r)$ is an $L$–space if $r \geq 2g(K) - 1$ or $r \leq -2g(K) + 1$. (Ozsváth-Szabó)

In particular, each hyperbolic, $L$–space knot produce infinitely many hyperbolic $L$–spaces by Dehn surgery.
Question

Which knots are $L$–space knots?
Which knots are \( L \)-space knots?

\( K \) is an \( L \)-space knot

\[\Rightarrow\]
Question

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$K$ is an $L$–space knot

$\Rightarrow$

- The non-zero coefficients of the Alexander polynomial of $K$ are $\pm 1$ and alternate in sign. (Ozsváth-Szabó)
Question

Which knots are $L$–space knots?

$K$ is an $L$–space knot

$\Rightarrow$

- The non-zero coefficients of the Alexander polynomial of $K$ are $\pm 1$ and alternate in sign. (Ozsváth-Szabó)

- $K$ is a fibered knot. (Ni)
Since a lens space is an $L$–space, a knot with lens space surgery is an $L$–space knot.
Examples of $L$–space knots

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- **Trivial knot** $O$
- **Torus knot** $T_{p,q}$
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Since a lens space is an $L$–space, a knot with lens space surgery is an $L$–space knot.

- **Trivial knot** $O$

- **Torus knot** $T_{p,q}$

- **Berge knots**, which are conjectured to comprise all knots with lens space surgeries.
• Montesinos knots

\[ K: \text{Montesinos knot} \]

\[ K \text{ is an } L\text{-space knot} \]

\[ \iff \]

\[ K = T_{2, 2n+1}, P(-2, 3, 2n+1), \text{ where } 0 \leq n \in \mathbb{Z} \text{ (up to mirror image)}. \]
Question

What operations on knots can keep a property of being L–space knots?
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- Some “cabling” operations are such operations. (Hedden)
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- Some “cabling” operations are such operations. (Hedden)

More generally,

- Some “satellite” operations using 1–bridge braid patterns are also such operations. (Hom-Lidman-Vafaee)
Question

Given an $L$–space knot $K$, does there exist an unknotted circle $c$ such that twistings $K$ along $c$ produce an infinite family of $L$–space knots?
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Example

\[
P(-2, 3, 1) = T_{5,2}
\]

L-space knot
Question

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Example

\[
P(-2, 3, 1) = T_{5,2} \\
P(-2, 3, 3) = T_{4,3}
\]

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**Question**

Given an $L$–space knot $K$, does there exist an unknotted circle $c$ such that twistings $K$ along $c$ produce an infinite family of $L$–space knots?

**Example**

<table>
<thead>
<tr>
<th>$P(-2, 3, 1)$</th>
<th>$P(-2, 3, 3)$</th>
<th>$P(-2, 3, 5)$</th>
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<td>$= T_{5,2}$</td>
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$E$-space knot
Question

Given an $L$–space knot $K$, does there exist an unknotted circle $c$ such that twistings $K$ along $c$ produce an infinite family of $L$–space knots?

Example

![Diagram of knot twistings]

P(-2, 3, 1) = T_{5,2}

P(-2, 3, 3) = T_{4,3}

P(-2, 3, 5) = T_{5,3}

P(-2, 3, 7)

non L-space knot

L-space knot
**Question**

Given an $L$–space knot $K$, does there exist an unknotted circle $c$ such that twistings $K$ along $c$ produce an infinite family of $L$–space knots?

**Example**

\[
\begin{align*}
P(-2, 3, 1) &= T_{5,2} \\
P(-2, 3, 3) &= T_{4,3} \\
P(-2, 3, 5) &= T_{5,3} \\
P(-2, 3, 7) &= T_{6,3}
\end{align*}
\]

\(\{P(-2, 3, 2n + 1)\}_{n \geq 0}\) is a twisted family of $L$–space knots.
Our approach

\(K\): \(L\)-space knot with \textbf{Seifert surgery} \((K, m)\),
i.e. \(K(m)\) is a Seifert fiber space

Twisting along a "\textit{seifert}\" \(c\) for \((K, m)\)
Our approach

\( K: L\)-space knot with Seifert surgery \((K, m)\),
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Twisting along a “seiferter” \(c\) for \((K, m)\)

### seiferter

A knot \(c\) in \(S^3 - K\) is called a seiferter for a Seifert surgery \((K, m)\)
if \(c\):

- is unknotted in \(S^3\),
- becomes a Seifert fiber in \(K(m)\).
Our approach

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  \begin{itemize}
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  \end{itemize}
\end{itemize}

In the following we allow the fibration to be degenerate, i.e. it contains an exceptional fiber of index 0 as a degenerate fiber.
**“Inheritance” of seiferters**

\[ S^3 \]

- \( K = T_{3,2} \)
- \( K_n \)
- n-twisting along \( c \)
- Dehn surgery

\[ S^3 \]

- \( c \) is unknotted
- \( c \) is a fiber

\[ S^3 \]

- \( c \) is a fiber

K notations:
- \( c \) is a fiber
- \( -3,2^n \)
“Inheritance” of seiferters

\[ K = T_{-3,2} \]

\[ \text{c is unknotted} \]

\[ S^3 \]

\[ (K, m) \text{ is a Seifert surgery } \Rightarrow (K_n, m_n) \text{ is a Seifert surgery} \]
Let $c$ be a seifter for $(K, m)$. 
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**$L$–space seiferter**

A seiferter $c$ for $(K, m)$ is an $L$–space seiferter if $M_c$ is an $L$–space.
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$M_c$: the result of $\lambda$–surgery along $c \subset K(m)$

**L–space seiferter**

A seiferter $c$ for $(K,m)$ is an *L–space seiferter* if $M_c$ is an *L–space*.

**Remark:** “$M_c = \lim_{n \to \infty} K_n(m_n)$”.
We call \((K, m)\) an \textit{L–space surgery} if \(K(m)\) is an \textit{L–space}.
We call \((K, m)\) an \(L\)-space surgery if \(K(m)\) is an \(L\)-space.

**Theorem 1**

Let \(c\) be a seiferter for a small Seifert fibered surgery \((K, m)\).

\((K_n, m_n)\) is an \(L\)-space surgery for infinitely many integers \(n\).

\[\Leftrightarrow\]

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\(c\) is an \(L\)-space seiferter.

For instance, \((O, m)\) has infinitely many \textit{\(L\)-space seiferters} for each \(m\), and we have:
We call \((K, m)\) an \textit{L–space surgery} if \(K(m)\) is an \(L–space\).

\textbf{Theorem 1}

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For instance, \((O, m)\) has infinitely many \textit{L–space seiferters} for each \(m\), and we have:

\textbf{Theorem 2 (L–space twisted unknots)}

\textit{For the trivial knot} \(O\), \textit{we can take} infinitely many unknotted circles \(c\) \textit{so that for each} \(c\) \textit{the twisted family} \(\{K_{c,n}\}_{|n|>1}\) \textit{is a set of mutually distinct hyperbolic L–space knots}.
$T_{p,q}(pq)$ is a connected sum of two lens spaces, and it has a degenerate Seifert fibration:

\[ T_{p,q}(pq) = \text{lens } \# \text{lens} \]

\[ S^2 \]

\[ \text{degenerate fiber = index 0 fiber} \]
$T_{p,q}(pq)$ is a connected sum of two lens spaces, and it has a degenerate Seifert fibration:

Degenerate Seifert fibration of $T_{p,q}(pq)$ is NOT unique!
Example

$c$ becomes a degenerate fiber in $T_{3,2}(6)$, hence $c$ is a seiferter for $(T_{3,2}, 6)$. 
Theorem 3

Let $K$ be $T_{p,q}$ or $C_{p,q}(T_{r,s})$ ($p = qr s \pm 1$) and $c$ a seiferter for $(K, pq)$. We assume $p, q \geq 2$.

Then $K_n$ is an $L$–space knot for any $n \geq -1$.

Furthermore, if the linking number $l$ between $c$ and $K$ satisfies $l^2 \geq 2pq$, then $K_n$ is an $L$–space knot for all integers $n$. 
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Example

$c$ is a seiferter for $(T_{3,2}, 6)$ and the linking number between $c$ and $T_{3,2}$ is 5. Since $5^2 \geq 2 \cdot 3 \cdot 2 = 12$, $K_n$ is an $L$–space knot for all integers $n$. 
Idea of the proof
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“inheritance” of seiferters
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+ 

Classification of Seifert fibered $L$–spaces
$M$: Seifert fiber space over $S^2$
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- $M$ is an $L$–space $\iff M$ admits no horizontal foliation.

$(\Rightarrow$ Ozsváth and Szabó$)\quad (\Leftarrow$ Lisca and Stipsicz$)$
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- Eisenbud-Hirsh-Neumann, Jankins-Neumann and Naimi gave a necessary and sufficient condition for $M$ to carry a horizontal foliation using Seifert invariants.
Seifert fibered $L$–spaces

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Combining them, we obtain:

- A necessary and sufficient condition for $M$ to be an $L$–space using Seifert invariants.
Use this condition to solve:

**Problem**

Given an integer $b$ and rational numbers $0 < r_1 \leq r_2 < 1$, find rational numbers $-1 \leq r \leq 1$ such that $S^2(b, r_1, r_2, r)$ is an $L$–space.
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**Solution:**

$$S^2(b, r_1, r_2, r): L\text{-space}$$

for

- $b \geq 1$
- $b = 0$
- $b = -1$ with $r_1 + r_2 \geq 1$
- $b = -1$ with $r_1 + r_2 \leq 1$
- $b = -2$
- $b \leq -3$
Recall that $c$ is a seiferter for $(K, pq)$, where $K = T_{p,q}$ or $C_{p,q}(T_{r,s})$, and $K(pq)$ is a connected sum of two lens spaces.
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Case I: $c$ is a **non-degenerate** fiber.
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Case I: $c$ is a **non-degenerate** fiber.

- $c$ is not a regular fiber. (Deruelle-Miyazaki-M)

$$K(m)\#lens = lens \# lens$$

$K(n) = lens \# lens$ or a lens space ($n = S_2 S_1$).

Hence $K(n) = lens$ for any integer $n$. (Deruelle-Miyazaki-M)
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Case I: $c$ is a non-degenerate fiber.

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- $K_n(m_n)$ is a connected sum of two lens spaces or a lens space ($\neq S^2 \times S^1$).
Recall that \( c \) is a seiferter for \((K, pq)\), where \( K = T_{p,q} \) or \( C_{p,q}(T_{r,s}) \), and \( K(pq) \) is a connected sum of two lens spaces.

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  \[
  K(m) = \text{lens} \# \text{lens}\]

- \( K_n(m_n) \) is a connected sum of two lens spaces or a lens space \((\neq S^2 \times S^1)\).

Hence \( K_n(m_n) \) is an \( L \)–space for any integer \( n \).
Case II: $c$ is a **degenerate** fiber.
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\[
\begin{align*}
K(m) &= \text{lens} \# \text{lens} \\
S^2 &= S^2(b, r_1, r_2, \infty) \\
b &\in \mathbb{Z}, \quad 0 < r_i < 1
\end{align*}
\]
Case II: \( c \) is a **degenerate** fiber.

Let \((\mu, \lambda)\) be a preferred meridian-longitude pair of \( c \subset S^3 \).
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Let \((\mu, \lambda)\) be a preferred meridian-longitude pair of \( c \subset S^3 \).

\[
\mu = t \quad \text{and} \quad \lambda = -s - \beta t \quad \text{in} \quad H_1(\partial N(c)) \quad \text{for some} \quad \beta \in \mathbb{Z}.
\]
$n$–twist along $c$ $\iff$ $-1/n$–surgery along $c$
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Let $(\mu_n, \lambda_n)$ be a preferred meridian-longitude pair of $c_n$. 
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$\mu_n = \mu - n\lambda$ \quad and \quad $\lambda_n = \lambda$. 
$n$–twist along $c$ $\iff$ $-1/n$–surgery along $c$

Let $(\mu_n, \lambda_n)$ be a preferred meridian-longitude pair of $c_n$.

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Then we have:

$\mu_n = ns + (n\beta + 1)t$ and $\lambda_n = -s - \beta t$. 
$n$–twist along $c \iff -1/n$–surgery along $c$

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Hence

$$K_n(m_n) = S^2(b, r_1, r_2, (n\beta + 1)/n).$$
Note that $K_n(m_n) = S^2(b, r_1, r_2, (n\beta + 1)/n) = S^2(b + \beta, r_1, r_2, 1/n)$. 
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Note that $K_n(m_n) = S^2(b, r_1, r_2, (n\beta + 1)/n) = S^2(b + \beta, r_1, r_2, 1/n)$.

- $K_n(m_n)$ is an $L$–space for $n = 0, \pm 1$.

We divide into four cases:

1. $b + \beta \leq -3$ or $b + \beta \geq 1$
2. $b + \beta = -2$
3. $b + \beta = -1$
4. $b + \beta = 0$. 
Assume: \( b + \beta \leq -3 \) or \( b + \beta \geq 1 \)
Assume: \( b + \beta \leq -3 \) or \( b + \beta \geq 1 \)

Then \( K_n(m_n) = S^2(\beta, r_1, r_2, 1/n) \) is an \( L \)-space if \(-1 \leq 1/n \leq 1\).
Assume: $b + \beta \leq -3$ or $b + \beta \geq 1$

Then $K_n(m_n) = S^2(b + \beta, r_1, r_2, 1/n)$ is an $L$–space if $-1 \leq 1/n \leq 1$.

Thus $K_n(m_n)$ is an $L$–space if $n \neq 0$. 
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Thus \( K_n(m_n) \) is an \( L \)-space if \( n \neq 0 \).

Since \( K_0(m_0) \) is also an \( L \)-space, \( K_n(m_n) \) is an \( L \)-space for all integers \( n \).
Assume: \( b + \beta = -2 \)
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Then \( \exists \varepsilon > 0 \) s.t.

\[ K_n(m_n) = S^2(b + \beta, r_1, r_2, 1/n) \] is an \textit{L–space} if \( -1 \leq 1/n \leq \varepsilon \).
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$K_n(m_n) = S^2(b + \beta, r_1, r_2, 1/n)$ is an $L$–space if $-1 \leq 1/n \leq \epsilon$.

Thus $K_n(m_n)$ is an $L$–space if $n \leq -1$ or $n \geq 1/\epsilon$. 
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Then \( \exists \epsilon > 0 \) s.t.

\[ K_n(m_n) = S^2(b + \beta, r_1, r_2, 1/n) \text{ is an } L\text{-space if } -1 \leq 1/n \leq \epsilon. \]

Thus \( K_n(m_n) \) is an \( L\)-space if \( n \leq -1 \) or \( n \geq 1/\epsilon \).

Since \( K_0(m_0) \) and \( K_1(m_1) \) are also \( L\)-spaces, \( K_n(m_n) \) is an \( L\)-space if \( n \leq 1 \) or \( n \geq 1/\epsilon \).
Assume: $b + \beta = -1$

- If $r_1 + r_2 \geq 1$, then $K_n(m_n) = S^2(b + \beta, r_1, r_2, 1/n)$ is an $L$–space for any $n$ with $0 < 1/n \leq 1$, i.e. $n \geq 1$

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Since \( K_n(m_n) \) is an \( L \)-space for \( n = 0, \pm 1 \), \( K_n(m_n) \) is an \( L \)-space for \( n \geq -1 \) \( (r_1 + r_2 \geq 1) \), or \( n \leq 1 \) \( (r_1 + r_2 \leq 1) \)
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<th>$b + \beta$</th>
<th>$K_n : L$-space knot</th>
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<tr>
<td>$\leq -3, \ 1 \leq$</td>
<td>$\forall n$</td>
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<td>$n \leq 1, \ n \geq 1/\varepsilon$</td>
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| $-1$ | $r_1 + r_2 \geq 1 \quad n \geq -1$
| | $r_1 + r_2 \leq 1 \quad n \leq 1$ |
| $0$ | $n \geq -1, \ n \leq -1/\varepsilon$ |
Further *homological arguments* show
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- $p, q \geq 2 \implies b + \beta \neq -2$
Further **homological arguments** show

- $p, q \geq 2 \implies b + \beta \neq -2$ and
  - $b + \beta = -1 \implies r_1 + r_2 > 1$
Further **homological arguments** show

- $p, \ q \geq 2 \ \Rightarrow \ b + \beta \neq -2$ and
  - $b + \beta = -1 \ \Rightarrow \ r_1 + r_2 > 1$

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$K_n$ is an $L$–space knot for any integer $n$. 

Kimihiko Motegi

$L$–space knots and twisting operation

Knots and Low Dimensional Manifolds
Busan, Korea
25 August, 2014

/ 46
Further **homological arguments** show

- $p, q \geq 2 \implies b + \beta \neq -2$ and
  
  $b + \beta = -1 \implies r_1 + r_2 > 1$

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$K_n$ is an $L$–space knot for any integer $n \geq -1$. 
\( l^2 \geq 2pq \quad \Rightarrow \quad b + \beta \neq -1, \ 0 \)
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\( K_n \) is an \( L \)–space knot for all integers \( n \).
twisted torus knot $K(p, q; r, n)$

$2 \leq r \leq p+q$

$n$-twist along $c$

$r$ strands
Application – twisted torus knots

Vafaee studies $K(p, kp; r, n)$ with $p \geq 2$, $k \geq 1$, $0 < r < p$, $n > 0$ from a viewpoint of knot Floer homology.
Making use of \textit{seiferters} for \((T_{p,q}, pq)\), as an application of Theorem 3, we obtain:
Application – twisted torus knots

Making use of seiferters for $(T_{p,q}, pq)$, as an application of Theorem 3, we obtain:

Theorem 4 (L–space twisted torus knots)

The following twisted torus knots are L–space knots for all integers $n$.

- $K(p, q; p + q, n)$ with $p, q \geq 2$
- $K(3p + 1, 2p + 1; 4p + 1, n)$ with $p > 0$
- $K(3p + 2, 2p + 1; 4p + 3, n)$ with $p > 0$
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**Theorem 4 (L–space twisted torus knots)**

1. The following twisted torus knots are \textit{L–space knots} for \textit{all integers} \(n\).
   - \(K(p, q; p + q, n)\) with \(p, q \geq 2\)
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   - \(K(3p + 2, 2p + 1; 4p + 3, n)\) with \(p > 0\)

2. The following twisted torus knots are \textit{L–space knots} for \textit{any} \(n \geq -1\).
   - \(K(p, q; p - q, n)\) with \(p, q \geq 2\)
   - \(K(2p + 3, 2p + 1; 2p + 2, n)\) with \(p > 0\)
In particular,

**Corollary 5**

For any torus knot $T_{p,q}$, we can take an unknotted circle $c$ so that $n$–twist along $c$ converts $T_{p,q}$ into an $L$–space knot $K_n$ for all integers $n$.

Furthermore, $\{K_n\}_{|n|>3}$ is a set of mutually distinct hyperbolic $L$–space knots.
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**Corollary 5**

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As we mentioned, for the trivial knot we can take infinitely many such unknotted circles.
Application – twisted Berge knots
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- Most Berge’s lens space surgeries are “next to” lens # lens surgeries.
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\[(P(-2, 3, 7), 19)\]
• Most Berge’s lens space surgeries are “next to” lens ≠ lens surgeries.

(P(-2, 3, 7), 19)

lens space surgery
Most Berge’s lens space surgeries are “next to” lens surgery.

\[
\begin{align*}
&\text{lens surgery} & (T_{5,3}, 15) & \text{(-1)-twist} & (P(-2, 3, 7), 19) \\
&\text{lens surgery} & & & \text{lens space surgery}
\end{align*}
\]
Application – twisted Berge knots

- Most Berge’s lens space surgeries are “next to” lens $\#$ lens surgeries.

\[ \text{lens} \# \text{lens surgery} \]
\[ (T_{5,3}, 15) \]
\[ \rightarrow \]
\[ (-1)\text{-twist} \]
\[ (P(-2, 3, 7), 19) \]

\[ \text{lens space surgery} \]
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**Theorem 6**

For any hyperbolic Berge knot $K$, there is an unknotted circle $c$ such that $n$–twist along $c$ converts $K$ into an $L$–space knot $K_n$ for infinitely many integers $n$. 
Applying Hedden’s cabling construction, Baker and Moore prove:

For any integer \( N \), there is a non-hyperbolic \( L \)-space knot with tunnel number greater than \( N \).

and ask:

Question

Is there a hyperbolic, \( L \)-space knot with tunnel number greater than one?
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**Theorem 7**

There exist infinitely many hyperbolic, \( L \)–space knots with tunnel number greater than one.
• $c_a$ and $c_b$ are seiferters for $(T_{3,2}, 7)$ simultaneously.
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• $\{K_{n,0}\}$ and $\{K_{0,n}\}$ are sets of mutually distinct hyperbolic knots with tunnel number 2. (Eudave–Muñoz–Jasso–Miyazaki–M)
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\(K_{n,0}(196n + 71)\) and \(K_{0,n}(100n + 71)\) are Seifert fibered \(L\)-spaces, hence \(K_{n,0}\) and \(K_{0,n}\) are \(L\)-space knots.
Does there exist a hyperbolic, \( L \)-space knot with tunnel number greater than two?

Let \( K_n \) be a knot obtained from an \( L \)-space knot \( K \) by \( n \)-twist along an unknotted circle \( c \). If the twisted family \( f_{K_n} \) contains infinitely many \( L \)-space knots, then does \( K \) admit a Seifert surgery for which \( c \) is a seiferter?
• Does there exist a hyperbolic, $L$–space knot with tunnel number greater than two?
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