On the clasp number of a knot

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**Fact.**

Every knot $K \subset S^3$ bounds a clasp disk $D$.

$c(D) :=$ the number of clasps of a clasp disk $D$.

$c(K) := \min\{c(D) \mid D: \text{a clasp disk of } K\} :$ the clasp number of $K$. 

The clasp number of a knot
**Proposition.** [Shibuya '74]

\[ \forall K, \max\{g(K), u(K)\} \leq c(K). \]

**Remark.**

Most of prime knots up to 10 crossings satisfy the equality above.

**Question.**

\[ \exists K: \text{a prime knot s.t. } \max\{g(K), u(K)\} < c(K). \]
Main Theorem. [Kadokami–Kawamura]

\(K_n\): the knot as shown below \((n \in \mathbb{Z})\).

\(n\) is odd \(\implies\) \(K_n\) is prime & \(\max\{g(K_n), u(K_n)\} < c(K_n)\).

Outline of proof.

1. Confirm that \(g(K_n) = 2\) and \(u(K_n) \leq 2\) (and \(c(K_n) \leq 4\)).
2. Investigate about the primeness of \(K_n\).
3. Prove that \(n\) is odd \(\implies\) \(c(K_n) \geq 3\).
Proof of $n$ is odd $\Rightarrow c(K_n) \geq 3$

$\nabla_K(z) \in \mathbb{Z}[z]$: the Conway polynomial of a knot $K$.

**Lemma.** [Kadokami–K.]

Suppose that a knot $K$ has $\nabla_K(z) = 1 + a_2z^2 + a_4z^4$.

Then, $a_2 \equiv 2 \ (\text{mod} \ 4)$ and $a_4 \equiv 3 \ (\text{mod} \ 8) \Rightarrow c(K) \geq 3$.

**Proof.** ( $n$ is odd $\Rightarrow c(K_n) \geq 3$. )

The Conway polynomial of $K_n$ is as follows:

$$\nabla_{K_n}(z) = 1 + 2nz^2 - (4n + 1)z^4.$$  

If $n$ is odd, then $2n \equiv 2 \ (\text{mod} \ 4)$ and $-(4n + 1) \equiv 3 \ (\text{mod} \ 8)$.

Therefore, by Lemma we have $c(K_n) \geq 3$.  

$\square$
$c(K_n) \leq 4$

$K_n$: 

$2n - 1$ half twists