Logarithmic knot invariants and hyperbolic volume

Jun Murakami

Waseda University

August 26, 2014
Outline:

1. Introduction

2. Logarithmic invariant of knots in $S^3$

3. Logarithmic invariant of knots in a 3-manifolds

4. Further generalizations
1. Introduction

Hyperbolic geometry

Almost all leaves have the hyperbolic structure.
1. Introduction

Hyperbolic geometry

Photo by Toby Hudson
1. Introduction

Hyperbolic geometry

Crochet coral reef project

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A mathematically precise model of a hyperbolic pane by Dr. Diana Taimina.
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Jones polynomial $J_L(q)$

- Skein relation
  
  \[ q^{-2} V_{L_+}(q) - q^2 V_{L_-}(q) = (q - q^{-1}) V_{L_0}(q). \]
Jones polynomial $J_L(q)$

- **Skein relation**
  \[
  q^{-2} V_{L_+}(q) - q^2 V_{L_-}(q) = (q - q^{-1}) V_{L_0}(q).
  \]

- **Kauffman bracket**
  \[
  \langle \begin{array}{c}
    \begin{array}{c}
      \includegraphics[width=0.1\textwidth]{kauffman_bracket_diagram1}
    \end{array}
    \end{array}\rangle = q^{1/2} \left( \begin{array}{c}
    \begin{array}{c}
      \includegraphics[width=0.12\textwidth]{kauffman_bracket_diagram2}
    \end{array}
    \end{array}\right) + q^{-1/2} \left( \begin{array}{c}
    \begin{array}{c}
      \includegraphics[width=0.1\textwidth]{kauffman_bracket_diagram3}
    \end{array}
    \end{array}\right), \quad \langle \phi \rangle = 1,
  \]

  \[
  V_L(q) = (-q^{3/2})^{-\text{wr}(L)} \langle L \rangle.
  \]

  where $\text{wr}(L)$ is the number of the positive crossings minus the number of the negative crossings of $L$. 

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  \[
  \langle \begin{array}{c} 
  \v\v\v\v
  \end{array} \rangle = q^{1/2} \langle \begin{array}{c} \v\v\v\v
  \end{array} \rangle + q^{-1/2} \langle \begin{array}{c} \v\v\v\v
  \end{array} \rangle, \quad \langle \phi \rangle = 1,
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  where $\text{wr}(L)$ is the number of the positive crossings minus the number of the negative crossings of $L$.

- **$R$-matrix of $\mathcal{U}_q(\mathfrak{sl}_2}$**
  $V_L(q)$ can be defined from the quantum $R$-matrix of the quantized enveloping algebra (quantum group) $\mathcal{U}_q(\mathfrak{sl}_2)$.
Colored Jones invariant

- Birman-Wenzl idenpotent $f_n$

The colored Jones invariant of color $n_1, n_2, \cdots$ is defined as follows. Replace the $j$-th string of the knot by parallel $n_j$ strings and insert a element $f_{n_j}$, which is defined inductively by

$$f_{n+1} = \frac{[n]}{[n+1]} f_n f_{n-1} f_n$$

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$
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  $$ f_{n_j} : \quad \begin{array}{c}
    n+1 \\
    n
  \end{array} = \begin{array}{c}
    1 \\
    n
  \end{array} + \frac{[n]}{[n+1]} \begin{array}{c}
    1 \\
    n
  \end{array} \begin{array}{c}
    1 \\
    n-1
  \end{array}, $$

  $$ [n] = \frac{q^n - q^{-n}}{q - q^{-1}}. $$

- Property of $f_n$

  $f_n$ satisfies the following property.

  $$ \begin{array}{c}
    j \\
    i
  \end{array} = \begin{array}{c}
    i+j \\
    i
  \end{array} $$

  $$ L_{n_1, n_2, \cdots} : \quad \langle L_{n_1, n_2, \cdots} \rangle = \left( q^{\frac{3}{2}} \right) \wr (\widetilde{L}_{n_1, n_2, \cdots}) \langle L_{n_1, n_2, \cdots} \rangle. $$
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The colored Jones invariant of color $n_1, n_2, \cdots$ is defined as follows. Replace the $j$-th string of the knot by parallel $n_j$ strings and insert an element $f_{n_j}$, which is defined inductively by

$$f_{n+1} = f_n + \frac{[n]}{[n+1]} f_{n-1},$$

where

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

- Property of $f_n$

$f_n$ satisfies the following property.

$$V_{L_{n_1,n_2,\cdots}}^{n_1+1,n_2+1,\cdots}(q) = (q^{\frac{3}{2}})^{-\text{wr}(L_{n_1,n_2,\cdots})} \langle L_{n_1,n_2,\cdots} \rangle.$$

- $R$-matrix of higher dimensional irreducible representation of $U_q(sl_2)$. 

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Quantum invariants

Alexander polynomial comes from the fundamental group of the knot complement, but also comes from the skein relation. Jones polynomial as above. HOMFLY-PT polynomial also comes from the skein relation and related to $U_q(sl_N)$.

Colored versions of the above invariants by replacing the vector representation with the highest weight representation.

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Universal $U_q(sl_2)$ invariant for knots by R. Laurence and T. Ohtsuki.

Witten-Reshetikhin-Turaev invariant for 3-manifolds.

Turaev-Viro invariant for 3-manifolds.

Kirillov-Reshetikhin invariant for knotted trivalent graphs in $S^3$.

Yokota invariant for knotted multivalent graphs in $S^3$.

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Kashaev’s invariant

Let $L$ be a knot in $S^3$, $N$ be a positive integer. Kashaev introduced an invariant of knot $K_N(L)$ from the quantum dilogarithm function. He also found the following relation for simplest three hyperbolic knots.

**Conjecture (Kashaev)**

$$\lim_{N \to \infty} \frac{2\pi}{N} \log |K_N(L)| = \text{Vol}_{\text{hyp}}(S^3 \setminus L).$$
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- **Hypervolic volume** Hypervolic volume $\text{Vol}_{\text{hyp}}(M)$ is the sum of the volumes of the hyperbolic pieces of the 3-manifold $M$. For other geometric structures, their hyperbolic volumes are defined to be 0.
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- **Hyperbolic volume**  Hypervolic volume $\text{Vol}_{hyp}(M)$ is the sum of the volumes of the hyperbolic pieces of the 3-manifold $M$. For other geometric structures, their hyperbolic volumes are defined to be 0.

- **Quantum dilogarithm function**

  Dilogarithm function : $\text{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t} \, dt$

  It relates to the volume of ideal tetrahedron and satisfies the **pentagon relation**.

  Faddeev and Kashaev replace the integral by $q$-sumension so that the results still satisfy the pentagon relation.
**Volume conjecture (1)**

- **Kashaev’s three examples** \( \xi_N = e^{\pi i/N}, \quad (\xi_N)_k = \prod_{j=1}^{k} (1 - \xi_N^{2j}) \),

\[
\begin{align*}
4_1: & \quad K_N(4_1) = \sum_{k=0}^{N-1} |(\xi_N)_k|^2 \\
5_2: & \quad K_N(5_2) = \sum_{k \leq l} \xi_N^{-k(l+1)} \frac{(\xi_N)_l^2}{(\bar{\xi}_N)_k} \\
6_1: & \quad K_N(6_1) = \sum_{k+l \leq m} \xi_N^{(m-k-l)(m-k+1)} \frac{|(\xi_N)_m|^2}{(\xi_N)_k(\bar{\xi}_N)_l}
\end{align*}
\]

\[\xrightarrow{N \to \infty} \begin{array}{c}2.02988321\ldots \\
2.82812208\ldots \\
3.16396322\ldots \end{array}\]
Volume conjecture (1)

Kashaev’s three examples  \( \xi_N = e^{\pi i/N}, \ (\xi_N)_k = \prod_{j=1}^{k} (1 - \xi_{N}^{2j}), \)

\[ K_N(4_1) = \sum_{k=0}^{N-1} |(\xi_N)_k|^2 \quad \xrightarrow{N \to \infty} \quad 2.02988321\ldots \]

\[ K_N(5_2) = \sum_{k \leq l} \xi_N^{-k(l+1)} \frac{(\xi_N)_k^2}{(\xi_N)_l} \xrightarrow{N \to \infty} \quad 2.82812208\ldots \]

\[ K_N(6_1) = \sum_{k+l \leq m} \xi_N^{(m-k-l)(m-k+1)} \frac{|(\xi_N)_m|^2}{(\xi_N)_k(\xi_N)_l} \xrightarrow{N \to \infty} \quad 3.16396322\ldots \]

Relation to the colored Jones invariant

Theorem (H. Murakami-J.M.)

For a knot \( L, \ K_N(L) = V_L^N(\xi_N). \)
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  4
  1 :  
  
  $K_N(4_1) = \sum_{k=0}^{N-1} |(\xi_N)_k|^2$  $\rightarrow$  $2.02988321\ldots$
  
  5
  2 :  
  
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  6
  1 :  
  
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- **Relation to the colored Jones invariant**

  **Theorem (H.Murakami-J.M.)**
  
  *For a knot $L$, $K_N(L) = V_L^N(\xi_N)$.***

  **Conjecture (H.Murakami-J.M.-M.Okamoto-T.Takata-Y.Yokota)**
  
  $\lim_{N \rightarrow \infty} 2\pi \frac{1}{N} \log V_L^N(\xi_N) = \text{Vol}_{\text{hyp}}(S^3 \setminus L) + \sqrt{-1} \text{ CS}(S^3 \setminus L)$. 
Volume conjecture (1)

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4₁: \[ K_N(4₁) = \sum_{k=0}^{N-1} |(\xi_N)_k|^2 \quad \rightarrow \quad N \rightarrow \infty \quad 2.02988321 \ldots \]

5₂: \[ K_N(5₂) = \sum_{k \leq l} \xi_N^{-k(l+1)} \frac{(\xi_N)_l^2}{(\xi_N)_k} \quad \rightarrow \quad N \rightarrow \infty \quad 2.82812208 \ldots \]

6₁: \[ K_N(6₁) = \sum_{k+l \leq m} \xi_N^{(m-k-l)(m-k+1)} \frac{|(\xi_N)_m|^2}{(\xi_N)_k(\xi_N)_l} \quad \rightarrow \quad N \rightarrow \infty \quad 3.16396322 \ldots \]

- **Relation to the colored Jones invariant**

**Theorem (H. Murakami-J.M.)**

For a knot \( L \), \( K_N(L) = V_L^N(\xi_N) \).

**Conjecture (H. Murakami-J.M.-M. Okamoto-T. Takata-Y. Yokota)**

\[ \lim_{N \rightarrow \infty} \frac{2\pi}{N} \log V_L^N(\xi_N) = \text{Vol}_{\text{hyp}}(S^3 \setminus L) + \sqrt{-1} \ \text{CS}(S^3 \setminus L). \]

  There is a tetrahedral decomposition of \( S^3 \setminus L \) such that each term of \( K_N(L) \) (or \( V_L^N(\xi_N) \)) corresponds to a tetrahedron of the decomposition.
Optimistic limit [H. Murakami]

Apply the saddle point method formally as follows: Let $V$ be the function obtained by replacing the quantum factorials in a quantum $U_q(sl_2)$ invariant by dilogarithm functions. $V$ is called the volume potential function. Then one of the critical values of $V$ relates to the hyperbolic volume of the corresponding geometric object for almost all cases.
Volume conjecture (2)

- **Optimistic limit** [H. Murakami]
  
  Apply the saddle point method *formally* as follows: Let $V$ be the function obtained by replacing the quantum factorials in a quantum $U_q(sl_2)$ invariant by dilogarithm functions. $V$ is called the *volume potential function*. Then one of the critical values of $V$ relates to the hyperbolic volume of the corresponding geometric object for almost all cases.

  **Application:** Volume formula for a generic hyperbolic tetrahedron from quantum $6j$ symbols [J.M.-M.Yano].
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- **Deformation of $q$** [S.Gukov-H.Murakami, H.Murakami-Y.Yokota]
  
  Deform $q$ around $\xi_N$, then $\lim_{N \to \infty} \frac{2\pi}{N} \log V_L^N(q)$ relates to the complex volume of a deformation of the hyperbolic structure of $S^3 \setminus L$. 
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Deform $q$ around $\xi_N$, then $\lim_{N \to \infty} \frac{2\pi}{N} \log V^N_L(q)$ relates to the complex volume of a deformation of the hyperbolic structure of $S^3 \setminus L$.

Kashaev’s invariant for knotted graphs [F.Costantino]

For a knotted graph $G$, $\lim_{N \to \infty} \frac{2\pi}{N} \log \left. \frac{KR^N_G(q)}{q^N - q^{-N}} \right|_{q=\xi_N} = \text{Vol}_{hyp}(S^3 \setminus G)$

where $KR^N_G$ be the Kilirrov-Reshetikhin invariant of $G$ whose edges are all colored by $N$. 
Purpose

Construct quantum invariants with finite state whose actual limit always relates to the hyperbolic volumes of the corresponding geometric objects.
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  The parameter $q$ should be $\xi_N$. 

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- **Finite state**
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- **Hyperbolic volume**
  The optimistic limits of quantum invariants represent the hyperbolic volumes. So, a little modification of the known invariants may be good enough to construct such invariants.
Logarithmic invariant $\gamma_{N,s}(L)$ of a knot $L$ in $S^3$ \quad (0 \leq s \leq N)
Logarithmic invariant $\gamma_{N,s}(L)$ of a knot $L$ in $S^3$ $(0 \leq s \leq N)$

**Theorem**

\[
\begin{align*}
\gamma_{N,0}(L) &= \frac{1}{2N} \left. V_L^{2N}(q) \right|_{q=\xi_N}, \\
\gamma_{N,N}(L) &= \frac{1}{N} \left. V_L^N(q) \right|_{q=\xi_N}, \\
\gamma_{N,s}(L) &= \frac{\xi_N}{2N} \left. \frac{d}{dq} \left\{ 1 \right\} (V_s(L) + V_{2N-s}(L)) \right|_{q=\xi_N}
\end{align*}
\]

where $\left\{ n \right\} = q^n - q^{-n}$.
Logarithmic invariant $\gamma_{N,s}(L)$ of a knot $L$ in $S^3$ ($0 \leq s \leq N$)

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\gamma_{N,N}(L) &= \frac{1}{N} V_L^{N}(q) \bigg|_{q=\xi_N}, \\
\gamma_{N,s}(L) &= \frac{\xi_N}{2N} \frac{d}{dq} \{1\} (V_s(L) + V_{2N-s}(L)) \bigg|_{q=\xi_N} \\
&\text{where } \{n\} = q^n - q^{-n}.
\end{align*}
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**Conjecture**

\[
\gamma_{N,s_N}(L) \sim_{N \to \infty} \exp \left( \frac{N}{2\pi} \left( \text{Vol}_{\text{hyp}}(M_\alpha) + \sqrt{-1} \text{CS}(M_\alpha) \right) \right)
\]

where $M_\alpha$ be the cone manifold along $L$ with cone angle $\alpha$ and $\lim_{N \to \infty} \frac{s_N}{N} = \frac{\alpha}{2\pi}$. 
Theorem

\[ \gamma_{N,0}(L) = \frac{1}{2N} V_L^{2N}(q) \bigg|_{q=\xi_N}, \quad \gamma_{N,N}(L) = \frac{1}{N} V_L^N(q) \bigg|_{q=\xi_N}, \]

\[ \gamma_{N,s}(L) = \frac{\xi_N}{2N} \frac{d}{dq} \left\{ 1 \right\} (V_s(L) + V_{2N-s}(L)) \bigg|_{q=\xi_N} \]

where \( \{n\} = q^n - q^{-n} \).

Conjecture

\[ \gamma_{N,s_N}(L) \xrightarrow{N \to \infty} \exp \left( \frac{N}{2\pi} (\text{Vol}_{\text{hyp}}(M_\alpha) + \sqrt{-1} \text{CS}(M_\alpha)) \right) \]

where \( M_\alpha \) be the cone manifold along \( L \) with cone angle \( \alpha \) and \( \lim_{N \to \infty} \frac{s_N}{N} = \frac{\alpha}{2\pi} \).

Numerical computation for the figure-eight knot 4_1.

\[ N = 200 \]
Logarithmic invariant of knots in a 3-manifold

Let $M$ be a three manifold and $L$ be a knot in $M$. The logarithmic invariant of $L$ is defined by using the Hennings invariant and let $\gamma_{N,s}^{SO(3)}$ be its $SO(3)$ version. Then $\gamma_{N,N}^{SO(3)}$ is a generalization of Kashaev’s invariant $K_N$. 

Table:

<table>
<thead>
<tr>
<th>$\gamma_{N,N}^{SO(3)}(M; L)$</th>
<th>Value</th>
</tr>
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<tbody>
<tr>
<td>$2$</td>
<td>7.4804</td>
</tr>
<tr>
<td>$3$</td>
<td>7.77535</td>
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<tr>
<td>$4$</td>
<td>7.74245</td>
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<tr>
<td>$5$</td>
<td>7.74245</td>
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Logarithmic invariant of knots in a 3-manifold

Let $M$ be a three manifold and $L$ be a knot in $M$. The logarithmic invariant of $L$ is defined by using the Hennings invariant and let $\gamma^{SO(3)}_{N,s}$ be its $SO(3)$ version. Then $\gamma^{SO(3)}_{N,N}$ is a generalization of Kashaev’s invariant $K_N$.

Let $M$ be a three manifold obtained from $f$ surgery around a component of the Whitehead link and $L$ be the knot represented by another component. Then $\gamma^{SO(3)}_{N,s}(M,L)$ is computed numerically as follows. ($M$ is a lens space.)

![Whitehead link](image)

<table>
<thead>
<tr>
<th>$f \setminus N$</th>
<th>83</th>
<th>123</th>
<th>183</th>
<th>245</th>
<th>$\text{Vol}_{\text{hyp}}$</th>
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<tbody>
<tr>
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<td>2.18415</td>
<td>2.13335</td>
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<td>2.02988</td>
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<td>2.02988</td>
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<td>3.45103</td>
<td>3.40030</td>
<td>3.37406</td>
<td>3.29690</td>
</tr>
</tbody>
</table>

Table: Values of $\pi \log \left| \gamma^{SO(3)}_{N,N}(M,L)/\gamma^{SO(3)}_{N-2,N-2}(M,L) \right|$
2. Logarithmic invariant of knots in $S^3$
2. Logarithmic invariant of knots in $S^3$

Restricted (small) quantum group

\[
\text{Colored Jones invariant } U_{q}(\mathfrak{sl}_2) = \langle K; E; F \rangle_{1} = q E; K F \rangle_{1} = q F; E F \rangle_{1} = K; K \rangle_{1} q^{-1} q^{-1} \rangle
\]

Restricted quantum group $U_{N}(\mathfrak{sl}_2)$

\[
\text{Logarithmic invariant } U_{N}(\mathfrak{sl}_2) = \langle K; E; F \rangle_{1} = N E; K F \rangle_{1} = q^2 F; E F \rangle_{1} = K; K \rangle_{1} q^{-1} q^{-1} \rangle
\]

Hopf algebra structure:

\[
\text{Coproduct: } \Delta(K) = K K, \Delta(E) = E K + 1 E, \Delta(F) = F + K F
\]

\[
\text{Counit: } \epsilon(K) = 1, \epsilon(E) = 0, \epsilon(F) = 0
\]

\[
\text{Antipode: } S(K) = K \pm 1, S(E) = E K, S(F) = K F
\]
2. Logarithmic invariant of knots in $S^3$

**Restricted (small) quantum group**

(Usual) quantum group $\mathcal{U}_q(sl_2)$

$$\mathcal{U}_q(sl_2) = \left\langle K, E, F \mid K E K^{-1} = q^2 E, \quad K F K^{-1} = q^{-2} F, \quad E F - F E = \frac{K - K^{-1}}{q - q^{-1}} \right\rangle$$

Colored Jones invariant
2. Logarithmic invariant of knots in $S^3$

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\]

Colored Jones invariant

Logarithmic invariant

Restricted quantum group $\overline{\mathcal{U}}_{\xi}(sl_2)$

\[
\overline{\mathcal{U}}_{\xi}(sl_2) = \langle K, E, F \mid K E K^{-1} = \xi_N^2 E, \quad K F K^{-1} = \xi_N q^{-2} F, \quad E F - F E = \frac{K - K^{-1}}{\xi_N - \xi_N^{-1}}, \quad E^N = F^N = 0, \quad K^{2N} = 1 \rangle
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2. Logarithmic invariant of knots in $S^3$

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**Logarithmic invariant**

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**Hopf algebra structure:**
2. Logarithmic invariant of knots in $S^3$

**Restricted (small) quantum group**

*(Usual) quantum group* $U_q(sl_2)$  

$$U_q(sl_2) = \left\langle K, E, F \mid KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \right\rangle$$

**Colored Jones invariant**

$U_q(sl_2)$

**Restricted quantum group** $\overline{U}_{\xi_N}(sl_2)$  

$$\overline{U}_{\xi_N}(sl_2) = \left\langle K, E, F \mid KEK^{-1} = \xi_N^2 E, \quad KFK^{-1} = \xi_N q^{-2} F, \quad EF - FE = \frac{K - K^{-1}}{\xi_N - \xi_N^{-1}}, \quad E^N = F^N = 0, \quad K^{2N} = 1 \right\rangle$$

**Logarithmic invariant**

Hopf algebra structure:

**Coproduct:**  

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \quad \Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F.$$
2. Logarithmic invariant of knots in $S^3$

**Restricted (small) quantum group**

(Usual) quantum group $\mathcal{U}_q(sl_2)$

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\mathcal{U}_q(sl_2) = \left\langle K, E, F \mid K E K^{-1} = q^2 E, \ K F K^{-1} = q^{-2} F, \ E F - F E = \frac{K - K^{-1}}{q - q^{-1}} \right\rangle
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\overline{\mathcal{U}}_{\xi_N}(sl_2) = \left\langle K, E, F \mid K E K^{-1} = \xi_N^2 E, \ K F K^{-1} = \xi_N q^{-2} F, \ E F - F E = \frac{K - K^{-1}}{\xi_N - \xi_N^{-1}}, \ E^N = F^N = 0, \ K^{2N} = 1 \right\rangle
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**Restricted quantum group $\overline{\mathcal{U}}_{\xi_N}(sl_2)$**

Logarithmic invariant

$$\overline{\mathcal{U}}_{\xi_N}(sl_2) = \left\langle K, E, F \mid K E K^{-1} = \xi_N^2 E, \quad K F K^{-1} = \xi_N q^{-2} F, \quad \frac{E F - F E}{\xi_N - \xi_N^{-1}} = \frac{K - K^{-1}}{\xi_N - \xi_N^{-1}}, \quad E^N = F^N = 0, \quad K^{2N} = 1 \right\rangle$$

**Hopf algebra structure:**

**Coproduct:**

$$\Delta(K^\pm 1) = K^\pm 1 \otimes K^\pm 1, \quad \Delta(E) = E \otimes K + 1 \otimes E,$$

$$\Delta(F) = F \otimes 1 + K^{-1} \otimes F.$$

**Counit:**

$$\epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0.$$

**Antipode:**

$$S(K^\pm 1) = K^{\mp 1}, \quad S(E) = -EK^{-1}, \quad S(F) = -KF.$$
Universal invariant

$L$: knot

$T$: tangle corresponding to $L$

$z_T$: the element of $\overline{U}_{\xi_N}$ obtained by the product of 'beads', where 'beads' are elements coming from the universal $R$ matrices associated to the crossings and $K^{\pm N\mp 1}$ at maximal and minimal points.
Universal invariant

$L$: knot

$T$: tangle corresponding to $L$

$z_T$: the element of $\mathcal{U}_N$ obtained by the product of 'beads', where 'beads' are elements coming from the universal $R$ matrices associated to the crossings and $K^{\pm N \mp 1}$ at maximal and minimal points.

Theorem (R. Lawrence, T. Ohtsuki, tangle version of the universal invariant)

*The element $z_T$ is an invariant of the knot $L$.***
Universal invariant

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Theorem (R. Lawrence, T. Ohtsuki, tangle version of the universal invariant)

The element $z_T$ is an invariant of the knot $L$.

Remark

The element $z_T$ is a center of $\overline{U}_{\xi_N}(sl_2)$, i.e. $z_T x = x z_T$ for any $x \in \overline{U}_{\xi_N}(sl_2)$. 
Centers and symmetric linear functions of $\mathcal{U}_{\xi N}(sl_2)$
Centers and symmetric linear functions of $\overline{U}_{\xi_N}(sl_2)$

- **Structure** of the indecomposable representation $P_s$
Centers and symmetric linear functions of $\overline{U}_{\xi_N}(sl_2)$

- **Structure** of the indecomposable representation $P_s^\pm$

\[
\begin{align*}
\chi^+_s(b_n) & \quad \chi^-_s(a_n) \\
\chi^\pm_N(y_k) & \quad \chi^\pm_{N-s}(x_k) \\
\chi^+_s(x_n) & : s \text{ dim. irred. repr. with basis } x_1, \cdots, x_s.
\end{align*}
\]

$\chi^+_1(K) \to -1, E \to 0, F \to 0$,

$\chi^-_1(x_n) = \chi^+_s \otimes \chi^-_1(x_n)$. 

- $E$ and $F$ are the nilpotent and central idempotent elements, respectively.

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Centers and symmetric linear functions of $\overline{\mathcal{U}}_{\xi N}(\mathfrak{sl}_2)$

- **Structure** of the indecomposable representation $\mathcal{P}_s^\pm$

  $$\begin{align*}
  &\mathcal{X}_s^{\pm}(b_n) & \xrightarrow{E} & \mathcal{X}_s^{\pm}(y_k) \\
  \downarrow w_s & \quad \mathcal{X}^\mp_{N-s}(x_k) & \xrightarrow{F} & \mathcal{X}^\mp_{N-s}(y_k) \\
  \xrightarrow{F} & \quad \mathcal{X}_s^{\pm}(a_n) & \xleftarrow{E}
  \end{align*}$$

  $\chi_s^{\pm}(x_n)$: $s$ dim. irred. repr. with basis $x_1, \cdots, x_s$.

  $\chi_1^-: K \to -1$, $E \to 0$, $F \to 0$,

  $\chi_s^-(x_n) = \chi_s^+ \otimes \chi_1^-(x_n)$.

- **Structure of the center**:

  $\psi: \psi(x) = (K^1)^{-1} N x$. 

  Right integral which satisfies $(\psi)(\Delta(x)) = (x) 1$.

  Modified right integral.
Centers and symmetric linear functions of $\overline{U}_{\xi_N}(sl_2)$

- **Structure** of the indecomposable representation $\mathcal{P}_s^\pm$

\[ \chi_s^\pm(b_n) \quad \xrightarrow{E} \quad \chi_s^\pm(a_n) \]

\[ \chi_{N-s}^\mp(x_k) \quad \xrightarrow{F} \quad \chi_{N-s}^\mp(y_k) \]

\[ \Downarrow w_s^\pm \]

$\chi_s^+(x_n) : s$-dim. irreducible representation with basis $x_1, \cdots, x_s$.

$\chi_s^-(x_n) = \chi_s^+ \otimes \chi_1^-(x_n)$.

- **Structure of the center:**

  $e_s :$ central idempotent (identity on $\chi_s^\pm$ and 0 on $\chi_t^\pm$, $t \neq s$)
Centers and symmetric linear functions of $\overline{U}_{\xi_N}(sl_2)$

**Structure** of the indecomposable representation $\mathcal{P}_s^{\pm}$

$$
\begin{align*}
\chi_s^{\pm}(b_n) & \quad \chi_s^{\pm}(x_n) : s \text{ dim. irred. repr. with basis } x_1, \cdots, x_s. \\
\mathcal{X}^{\mp}_{N-s}(x_k) & \quad \chi_1^- : K \rightarrow -1, \ E \rightarrow 0, \ F \rightarrow 0, \\
\mathcal{X}^{\mp}_{N-s}(y_k) & \quad \chi_s^-(x_n) = \chi_s^{+} \otimes \chi_1^-(x_n).
\end{align*}
$$

**Structure of the center:**

$e_s :$ central idempotent (identity on $\mathcal{X}_s^{\pm}$ and 0 on $\mathcal{X}_t^{\pm}$, $t \neq s$)

$w_s^{\pm}$: nilpotent element which maps $\mathcal{X}_s^{\pm}(b_n)$ to $\mathcal{X}_s^{\pm}(a_n)$ in $\mathcal{P}_s^{\pm}.$
Centers and symmetric linear functions of $\overline{U}_{\xi_N}(sl_2)$

- **Structure** of the indecomposable representation $\mathcal{P}_s^\pm$

  $\begin{array}{c}
  \chi^+_s(b_n) \\
  \downarrow E \\
  \chi_s^\mp(x_k) \\
  \downarrow w_s^\pm \\
  \chi_s^\mp(a_n)
  \end{array}$

  $\begin{array}{c}
  \chi_s^\pm(x_n) : s \text{ dim. irred. repr. with basis } x_1, \cdots, x_s. \\
  \chi^+_1 : K \rightarrow -1, \quad E \rightarrow 0, \quad F \rightarrow 0, \\
  \chi^-_s(x_n) = \chi^+_s \otimes \chi^-_1(x_n).
  \end{array}$

- **Structure of the center:**
  
  $e_s :$ central idempotent (identity on $\chi^\pm_s$ and 0 on $\chi^\pm_t$, $t \neq s$)

  $w_s^\pm :$ nilpotent element which maps $\chi^\pm_s(b_n)$ to $\chi^\pm_s(a_n)$ in $\mathcal{P}_s^\pm$.

- **Symmetric linear functions:** $f(xy) = f(yx)$
Centers and symmetric linear functions of $\mathcal{U}_{\xi_N}(sl_2)$

- **Structure** of the indecomposable representation $\mathcal{P}_s^\pm$

  $\chi_s^\pm(b_n) \quad \xrightarrow{E} \quad \chi_s^+(x_n)$

  $\chi_s^\mp(\cdot) : s$ dim. irred. repr. with basis $x_1, \ldots, x_s$.

  $\chi_s^-(x_n) = \chi_s^\pm \otimes \chi_1^-(x_n).$

- **Structure of the center:**

  $e_s :$ central idempotent (identity on $\chi_s^\pm$ and 0 on $\chi_t^\pm, t \neq s$)

  $w_s^\pm :$ nilpotent element which maps $\chi_s^\pm(b_n)$ to $\chi_s^\pm(a_n)$ in $\mathcal{P}_s^\pm$.

- **Symmetric linear functions:**

  $f(xy) = f(yx)$

  $X_s :$ trace of $\chi_s$

---

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Centers and symmetric linear functions of $U_{\xi_N}(sl_2)$

**Structure** of the indecomposable representation $P_s^\pm$

- $\chi_s^\pm(b_n)$
- $\chi_s^\pm(a_n)$
- $\chi_s^\pm(x_n)$: $s$ dim. irred. repr. with basis $x_1, \cdots, x_s$.
- $\chi_1^- : K \rightarrow -1$, $E \rightarrow 0$, $F \rightarrow 0$,
  \[ \chi_s^-(x_n) = \chi_s^+ \otimes \chi_1^-(x_n). \]

**Structure of the center:**

- $e_s$ : central idempotent (identity on $\chi_s^\pm$ and 0 on $\chi_t^\pm$, $t \neq s$)
- $w_s^\pm$: nilpotent element which maps $\chi_s^\pm(b_n)$ to $\chi_s^\pm(a_n)$ in $P_s^\pm$.

**Symmetric linear functions:**

- $f(xy) = f(yx)$
- $X_s$ : trace of $\chi_s$
- $G_s$ : trace of the block corresponding to $\chi_s^\pm(a_m)$ and $\chi_s^\pm(b_m)$. 

Centers and symmetric linear functions of $\overline{U}_{\xi_N}(sl_2)$

- **Structure** of the indecomposable representation $P^\pm_s$

  \[
  \begin{align*}
  \chi^+_s(b_n) & \quad \chi^+_s(x_n) : s \text{ dim. irred. repr. with basis } x_1, \ldots, x_s. \\
  \chi^+_s(y_k) & \quad \chi^-_1 : K \to -1, \quad E \to 0, \quad F \to 0, \\
  \chi^-_s(x_n) = \chi^+_s \otimes \chi^-_1(x_n).
  \end{align*}
  \]

- **Structure of the center:**

  \[e_s : \text{central idempotent (identity on } \chi^\pm_s \text{ and 0 on } \chi^\pm_t, \ t \neq s)\]

  \[w^\pm_s : \text{nilpotent element which maps } \chi^\pm_s(b_n) \text{ to } \chi^\pm_s(a_n) \text{ in } P^\pm_s.\]

- **Symmetric linear functions:**

  \[f(xy) = f(yx)\]

  \[X_s : \text{trace of } \chi_s\]

  \[G_s : \text{trace of the block corresponding to } \chi^\pm_s(a_m) \text{ and } \chi^\pm_s(b_m).\]

  \[\mu : \text{right integral which satisfies } (\mu \otimes \text{id})(\Delta(x)) = \mu(x) 1.\]
Centers and symmetric linear functions of $\overline{U}_{\xi_N}(sl_2)$

- **Structure** of the indecomposable representation $\mathcal{P}_s^\pm$

  \[ \begin{array}{ccc}
  \chi_s^\pm(b_n) & \xrightarrow{E} & \chi_s^\pm(a_n) \\
  \downarrow w_s^\pm & & \downarrow w_s^\pm \\
  \chi_{N-s}^\mp(x_k) & \xrightarrow{F} & \chi_{N-s}^\mp(y_k)
  \end{array} \]

  $\chi_s^\pm(x_n) : s$ dim. irred. repr. with basis $x_1, \cdots, x_s$.

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- **Symmetric linear functions:**

  $f(xy) = f(yx)$

  $X_s :$ trace of $\chi_s$

  $G_s :$ trace of the block corresponding to $\chi_s^\pm(a_m)$ and $\chi_s^\pm(b_m)$.

  $\mu :$ **right integral** which satisfies $(\mu \otimes \text{id})(\Delta(x)) = \mu(x) 1$.

  $\phi : \phi(x) = \mu(K^{1-N} x)$. **modified right integral**
$SL(2, \mathbb{Z})$ action on the center
$SL(2, \mathbb{Z})$ action on the center

generators

\[ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]

\[ S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]
$SL(2, \mathbb{Z})$ action on the center

generators

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\phi$: the symmetric linear function corresponding to the right integral
$SL(2, \mathbb{Z})$ action on the center

Generators

\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

\[\phi: \text{the symmetric linear function corresponding to the right integral}\]

Good basis of the center of $\overline{U}_q(sl_2)$

\[
\hat{\kappa}(0) = e_0, \quad \hat{\kappa}(s) = \frac{1}{[s]^2} \left( w_s^+ + w_{N-s}^- \right), \quad \hat{\kappa}(N) = -e_N,
\]

\[
\hat{\rho}(s) = (-1)^{N+s} \frac{1}{N(q^s-q^{-s})} \left( e_s - \frac{q^s+q^{-s}}{[s]^2} \left( w_s^+ + w_{N-s}^- \right) \right)
\]

\[
\hat{\varphi}(s) = \frac{1}{[s]^2} \left( \frac{N-s}{N} w_s^+ - \frac{S}{N} w_{N-s}^- \right) \quad (1 \leq s \leq N-1)
\]

Structure of $\mathcal{P}_s^\pm$

\[
\begin{array}{c}
\chi_s^\pm(b_n) \\
\chi_{N-s}^\pm(x_k) \downarrow w_s^\pm \downarrow \chi_{N-s}^\pm(y_k) \\
\chi_s^\pm(a_n)
\end{array}
\]

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Structure of $SL(2, \mathbb{Z})$ representation
(B.Feigin, A.Gainutdinov, A.Semikhatov, I.Tipunin)
It comes from the logarithmic Conformal Field Theory.
Structure of $SL(2, \mathbb{Z})$ representation

(B. Feigin, A. Gainutdinov, A. Semikhatov, I. Tipunin)

It comes from the logarithmic Conformal Field Theory.

$\hat{\kappa}(s), \ (\hat{\rho}(s), \hat{\varphi}(s))$

$\mathbb{Z} \cong \mathbb{C}^{N+1} \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{N-1})$

**Volume** $H_1, \text{ WRT}$
Structure of $SL(2, \mathbb{Z})$ representation  
(B.Feigin, A.Gainutdinov, A.Semikhatov, I.Tipunin)  
It comes from the logarithmic Conformal Field Theory.

$\hat{\kappa}(s), \ (\hat{\rho}(s), \hat{\varphi}(s))$

$\mathcal{Z} \cong \mathbb{C}^{N+1} \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{N-1})$

Volume $H_1, \ WRT$

$\lambda_{N,s} = q^{\frac{s^2-1}{2}}$

$T \hat{\kappa}(s) = \lambda_{N,s} \hat{\kappa}(s), \ T \hat{\varphi}(s) = \lambda_{N,s} \hat{\varphi}(s), \ T \hat{\rho}(s) = \lambda_{N,s} (\hat{\rho}(s) + \hat{\varphi}(s)),$

$S \hat{\kappa}(s) = \frac{1}{\sqrt{2N}} \left( (-1)^{N-s} \hat{\kappa}(0) + \sum_{t=1}^{N-1} (-1)^{N+t+s} (q^{st} + q^{-st}) \hat{\kappa}(t) + \hat{\kappa}(N) \right),$

$S \hat{\rho}(s) = \frac{1}{\sqrt{2N}} \sum_{t=1}^{N-1} (-1)^{N+t+s} (q^{st} - q^{-st}) \hat{\varphi}(t),$

$S \hat{\varphi}(s) = \frac{1}{\sqrt{2N}} \sum_{t=1}^{N-1} (-1)^{N+t+s} (q^{st} - q^{-st}) \hat{\rho}(t).$

---

$\hat{\kappa}(0) = (-1)^{N+1} e_0, \ \hat{\kappa}(s) = \frac{1}{[s]^2} \left( w^+_s + w^-_{N-s} \right), \ \hat{\kappa}(N) = e_N$

$\hat{\rho}(s) = \frac{(-1)^{N+s}}{N(q^s - q^{-s})} \left( e_s - \frac{q^s + q^{-s}}{[s]^2} (w^+_s + w^-_{N-s}) \right), \ \hat{\varphi}(s) = \frac{1}{[s]^2} \left( \frac{N-s}{N} w^+_s - \frac{s}{N} w^-_{N-s} \right)$
Coefficients

Let

\[ z_L = \sum_{s=1}^{N-1} \alpha_{N,s}(L) \hat{\rho}_s + \sum_{s=1}^{N-1} \beta_{N,s}(L) \hat{\varphi}_s + \sum_{s=0}^{N} \gamma_{N,s}(L) \hat{\kappa}_s. \]
Coefficients

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Theorem

\[ \gamma_{N,0}(L) = \left. \frac{V_L^{2N}(q)}{[2N]} \right|_{q=\xi_N} = \frac{\xi_N}{4N} \frac{d}{dq} \{1\}_q V_L^{2N}(q) \bigg|_{q=\xi_N} = \left. \frac{N \{1\}}{2 \pi \sqrt{-1}} \frac{d}{dm} V_L^m(q) \right|_{m=2N}, \]

\[ \gamma_{N,s}(L) = \frac{\xi_N}{2N} \frac{d}{dq} \{1\}_q (V_L^s(q) + V_L^{2N-s}(q)) \bigg|_{q=\xi_N} = \left. \frac{N \{1\}}{\pi \sqrt{-1}} \frac{d}{dm} V_L^m(q) \right|_{m=s}, \]

\[ 1 \leq s \leq N - 1, \]

\[ \gamma_{N,N}(L) = \left. -\frac{V_L^N(q)}{[N]} \right|_{q=\xi_N} = \frac{\xi_N}{2N} \frac{d}{dq} \{1\}_q V_L^N(q) \bigg|_{q=\xi_N} = \left. \frac{N \{1\}}{2 \pi \sqrt{-1}} \frac{d}{dm} V_L^m(q) \right|_{m=N}. \]
Coefficients

Let

\[ z_L = \sum_{s=1}^{N-1} \alpha_{N,s}(L) \hat{\rho}_s + \sum_{s=1}^{N-1} \beta_{N,s}(L) \hat{\varphi}_s + \sum_{s=0}^{N} \gamma_{N,s}(L) \hat{k}_s. \]

\[ = \frac{V_L^{2N}(q)}{2N} \bigg|_{q=\xi_N} - \frac{V_L^{N}(q)}{N} \bigg|_{q=\xi_N} \]

\[ = \frac{\xi_N}{2N} \frac{d}{dq} \{1\} q V_L^{2N}(q) \bigg|_{q=\xi_N} = \frac{N \{1\}}{2 \pi \sqrt{-1}} \frac{d}{dm} V_L^{m}(q) \bigg|_{m=2N, q=\xi_N} \]

\[ = \frac{\xi_N}{2N} \frac{d}{dq} \{1\} q V_L^{N}(q) \bigg|_{q=\xi_N} = \frac{N \{1\}}{2 \pi \sqrt{-1}} \frac{d}{dm} V_L^{m}(q) \bigg|_{m=N, q=\xi_N}. \]

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\[ 1 \leq s \leq N - 1, \]

\[ \gamma_{N,N}(L) = - \frac{V_L^{N}(q)}{N} \bigg|_{q=\xi_N} = \frac{\xi_N}{2N} \frac{d}{dq} \{1\} q V_L^{N}(q) \bigg|_{q=\xi_N} = \frac{N \{1\}}{2 \pi \sqrt{-1}} \frac{d}{dm} V_L^{m}(q) \bigg|_{m=N, q=\xi_N}. \]

Habiro’s universal formula for the colored Jones invariant

\[ V_L^{m}(q) = \sum_{k=0}^{\infty} a_k(L) \frac{\{m+k\} \{m+k-1\} \cdots \{m-k\}}{\{1\}} \]

where \( a_k(L) \) is a Laurent polynomial in \( q \) which does not depend on \( m \).
Relation to the hyperbolic volume
Relation to the hyperbolic volume

Conjecture

\[ \gamma_{N,s} (L) \underset{N \to \infty}{\sim} \exp \left( \frac{N}{2 \pi} \left( \text{Vol}_{\text{hyp}} (M_\alpha) + \sqrt{-1} \text{CS} (M_\alpha) \right) \right) \]

where \( M_\alpha \) be the cone manifold along \( L \) with cone angle \( \alpha \) and \( \lim_{N \to \infty} \frac{s_N}{N} = \frac{\alpha}{2\pi} \).
Relation to the hyperbolic volume

**Conjecture**

\[
\gamma_{N,s_N}(L) \xrightarrow{N \to \infty} \exp\left(\frac{N}{2\pi} \left(\text{Vol}_{\text{hyp}}(M_\alpha) + \sqrt{-1} \text{CS}(M_\alpha)\right)\right)
\]

where \( M_\alpha \) be the cone manifold along \( L \) with cone angle \( \alpha \) and \( \lim_{N \to \infty} \frac{s_N}{N} = \frac{\alpha}{2\pi} \).

Numerical computation for the figure-eight knot \( 4_1 \).
3. Logarithmic invariant of knots in a 3-manifolds
3. Logarithmic invariant of knots in a 3-manifolds

**Universal invariant of** $\overline{U}_q(sl_2)$

**Construction**

$L = L_1 \cup \cdots \cup L_k$ : framed link

$u(L) = \sum_j u_1^{(j)} \otimes u_2^{(j)} \otimes \cdots \otimes u_k^{(j)} \in \overline{U}_{\xi_N}(sl_2) \otimes \overline{U}_{\xi_N}(sl_2) \otimes \cdots \otimes \overline{U}_{\xi_N}(sl_2)$.

$f_i \in \left( \overline{U}_q(sl_2) / [\overline{U}_q(sl_2), \overline{U}_q(sl_2)] \right)^*$

**symmetric linear function**

R. Lawrence, T. Ohtsuki

Beads construction
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**Theorem**

$$\sum_{j} \prod_{i=1}^{k} f_i(u_i^{(j)})$$ is an invariant of $L$. 

R. Lawrence, T. Ohtsuki
Universal invariant of $\mathcal{U}_q(sl_2)$

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symmetric linear function

Theorem

$\sum_{j}^{k} \prod_{i=1}^{\lambda} f_i(u^{(j)}_i)$ is an invariant of $L$.

- **Colored Jones invariant** : $f_i$ is the trace on $\chi_s^+$ ($s$ dimensional irred. rep.)
3. Logarithmic invariant of knots in a 3-manifolds

Universal invariant of $\overline{U}_q(sl_2)$

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symmetric linear function

Theorem

$$\sum_{j} \prod_{i=1}^{k} f_i(u_i^{(j)}) \text{ is an invariant of } L.$$  

- Colored Jones invariant : $f_i$ is the trace on $\chi_s^+ (s \text{ dimensional irred. rep.})$
- Hennings invariant : $f_i = \phi$ (modified right integral)
Any 3-manifold is given by the surgery along a framed link in $S^3$. $L$, $L'$ represent the same manifold iff $L'$ is obtained from $L$ by a sequence of Kirby moves.

$$L \Leftrightarrow L'$$

Right integral:

$$\langle x \rangle_1 = m(\langle 1 \rangle)(\Delta(\langle x \rangle))$$

Recall that $\phi(x) = (K^N + 1)x$, $\phi(xy) = \phi(yx)$. 

**Hennings invariant**

Let $M$ be a 3-manifold given by $L$, $u(L)$ be its universal invariant, $\tilde{H}^N(L) = \sum_j \prod_{i=1}^{\# pos.(neg.) \text{ eigenvalues}} f_i(u(j))$ and $H^N(L) = \tilde{H}^N(L) \tilde{H}^N(L^{+1}) + \tilde{H}^N(L^{1})$ where $^+$ is the # of pos.(neg.) eigenvalues of the linking matrix of $L$.

**Theorem (M. Hennings)**

$H^N(L)$ is an invariant of $M$. 

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Hennings invariant

- Kirby move
  Any 3-manifold is given by the surgery along a framed link in $S^3$. $L, L'$ represent the same manifold iff $L'$ is obtained from $L$ by a sequence of Kirby moves.
Hennings invariant

- **Kirby move**
  Any 3-manifold is given by the surgery along a *framed link* in $S^3$. $L, L'$ represent the *same* manifold iff $L'$ is obtained from $L$ by a sequence of *Kirby moves*.

- **Right integral** $\lambda : \overline{U}_{\xi_N}(sl_2) \to \mathbb{C}$
  $\lambda$ satisfies $\lambda(x)1 = m(\lambda \otimes 1)(\Delta(x))$. Recall that $\phi(x) = \lambda(K^{N+1}x)$, $\phi(xy) = \phi(yx)$.
**Hennings invariant**

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  where $\sigma_+$ ($\sigma_-$) is the # of pos.(neg.) eigenvalues of the linking matrix of $L$. 

\[ \begin{array}{c}
\text{KI.} \\
\begin{array}{c}
L \\
\text{L}'
\end{array}
\end{array} \quad \leftrightarrow \quad \pm 1 \] 

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\text{KII.} \\
\begin{array}{c}
L \\
\text{L}'
\end{array}
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  Let $M$ be a 3 manifold given by $L$, $u(L)$ be its universal invariant, $\tilde{H}_N(L) = \sum_j \prod_{i=1}^k f_i(u_i^{(j)})$ and
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  where $\sigma_+$ ($\sigma_-$) is the # of pos.(neg.) eigenvalues of the linking matrix of $L$.

**Theorem (M. Hennings)**

$H_N(L)$ is an invariant of $M$. 
Let $L_1 \cup L \subset S^3$ be a framed link in $S^3$, $M$ be a 3-manifold obtained from $L_1 \cup L$, and $L$ be the knot in $M$ which is the image of $L_1$. Let $T$ be a tangle obtained from $L_1 \cup L$ by cutting $L_1$ and $u(T)$ be the universal invariant of $T$. Apply $\phi$ to the components of $u(T)$ corresponding to $L \subset M$, we get an element $z(L_1 \cup L) \in \mathbb{U}_N(\mathfrak{sl}_2)$.

Let $z(L_1 \cup L) = \sum_{s=1}^N z_{s}^{L_1 \cup L}$, $z_{s}^{L_1 \cup L}$ be invariant of $(M; L)$.

Remark $z_{s}^{L_1 \cup L}$, $z_{s}^{L_1 \cup L}$ are given by the WRT invariant.
Logarithmic invariant

Let $L_1 \cup L_M$ be a framed link in $S^3$, $M$ be a 3 manifold obtained from $L_M$, and $L$ be the knot in $M$ which is the image of $L_1$. Let $T$ be a tangle obtained from $L_1 \cup L_M$ by cutting $L_1$ and $u(T)$ be the universal invariant of $T$. Apply $\phi$ to the components of $u(T)$ corresponding to $L_M$, we get an element $z(L_1 \cup L_M) \in \overline{U}_{\xi_N}(sl_2)$.

Theorem

$N;\gamma(L_1 \cup L_M)$, $N;\gamma(L_1 \cup L_M)$, $N;\gamma(L_1 \cup L_M)$ are invariants of $(M; L)$.

Remark

$N;\gamma(L_1 \cup L_M)$, $N;\gamma(L_1 \cup L_M)$, are given by the WRT invariant. Let $N;\gamma(M; L) = N;\gamma(L_1 \cup L_M)$, which is called the logarithmic invariant of knot $L$ in $M$. Especially, $N;\gamma(M; L)$ is a generalization of Kashaev's invariant.
Logarithmic invariant

Let $L_1 \cup L_M$ be a framed link in $S^3$, $M$ be a 3 manifold obtained from $L_M$, and $L$ be the knot in $M$ which is the image of $L_1$. Let $T$ be a tangle obtained from $L_1 \cup L_M$ by cutting $L_1$ and $u(T)$ be the universal invariant of $T$. Apply $\phi$ to the components of $u(T)$ corresponding to $L_M$, we get an element $z(L_1 \cup L_M) \in \overline{U}_{\xi_N}(sl_2)$.

Let

$$z_{L_1 \cup L_M} = \sum_{s=1}^{N-1} \alpha_{N,s}(L_1 \cup L_M) \hat{\rho}_s + \sum_{s=1}^{N-1} \beta_{N,s}(L_1 \cup L_M) \hat{\phi}_s + \sum_{s=0}^{N} \gamma_{N,s}(L_1 \cup L_M) \hat{\kappa}_s.$$
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Let $L_1 \cup L_M$ be a framed link in $S^3$, $M$ be a 3 manifold obtained from $L_M$, and $L$ be the knot in $M$ which is the image of $L_1$. Let $T$ be a tangle obtained from $L_1 \cup L_M$ by cutting $L_1$ and $u(T)$ be the universal invariant of $T$. Apply $\phi$ to the components of $u(T)$ corresponding to $L_M$, we get an element $z(L_1 \cup L_M) \in \overline{U}_{\xi_N}(sl_2)$.

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Theorem

$\alpha_{N,s}(L_1 \cup L_M), \beta_{N,s}(L_1 \cup L_M), \gamma_{N,s}(L_1 \cup L_M)$ are invariants of $(M, L)$.
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Let $L_1 \cup L_M$ be a framed link in $S^3$, $M$ be a 3 manifold obtained from $L_M$, and $L$ be the knot in $M$ which is the image of $L_1$. Let $T$ be a tangle obtained from $L_1 \cup L_M$ by cutting $L_1$ and $u(T)$ be the universal invariant of $T$. Apply $\phi$ to the components of $u(T)$ corresponding to $L_M$, we get an element $z(L_1 \cup L_M) \in U_N(sl_2)$.

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\[
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\]

**Theorem**

$\alpha_{N,s}(L_1 \cup L_M), \beta_{N,s}(L_1 \cup L_M), \gamma_{N,s}(L_1 \cup L_M)$ are invariants of $(M, L)$.

**Remark**

$\alpha_{N,s}(L_1 \cup L_2), \beta_{N,s}(L_1 \cup L_2)$, are given by the WRT invariant.

Let $\gamma_{N,s}(M, L) = \gamma_{N,s}(L_1 \cup L_2)$, which is called the **logarithmic invariant** of knot $L$ in $M$. Especially, $\gamma_{N,N}(M, L)$ is a generalization of Kashaev’s invariant.
Relation to the hyperbolic volume

Let $N$ be a positive odd integer and $\phi^{SO(3)}$ be the $SO(3)$ version of $\phi$. $\phi$ is a linear combination of symmetric linear functions $X_s^\pm$, $G_s$ and $\phi^{SO(3)}$ is a sum over odd $s$. Then $\gamma_{N,s}^{SO(3)}(M,L)$ is constructed by replacing $\phi$ by $\phi^{SO(3)}$. 

Conjecture

$$\lim_{N \to 1} 2 \log_{SO(3)} N \gamma_{N,L}(M,L) = \text{Vol}_{hyp}(M,L) + \sqrt{1 - CS(M,L)}.$$
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**Conjecture**

$$\lim_{N \to \infty} \frac{2\pi \log \gamma^{SO(3)}_{N,N}(M, L)}{N} = \text{Vol}_{hyp}(M \setminus L) + \sqrt{-1} \text{ CS}(M \setminus L).$$

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**Table:** Values of $\pi \log \left| \frac{\gamma^{SO(3)}_{N,N}(M, L) / \gamma^{SO(3)}_{N-2,N-2}(M, L)}{N} \right|$

Whitehead link

Jun Murakami (Waseda University)  Logarithmic invariants  August 26, 2014  23 / 25
4. Further generalization

Logarithmic invariant of knotted graphs
It may be constructed by using the integral Kauffman bracket for trivalent graphs by F. Costantino.

Logarithmic invariant of closed 3-manifolds
It may be constructed by using K. Habiro's universal Witten-Reshetikhin-Turaev invariant.

Final remark
1. Logarithmic invariant cannot be obtained from any trace, conditional expectation nor symmetric linear function (pseudo trace). Here we use center instead of trace and its variants.
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Jun Murakami  (Waseda University) Logarithmic invariants August 26, 2014 24 / 25
THANK YOU!