On the number of contact isotopies of tight contact structures of the hyperbolic 3-manifolds

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Contact Manifold

Definition
Let $M$ be a $(2n+1)$-dimensional smooth manifold. If $M$ has a maximally nonintegrable hyperplane field $\xi$, then we call $(M, \xi)$ contact manifold. If $\xi$ is a coorientable contact hyperplane, then there is a global 1-form $\alpha$ such that $\xi = \ker \alpha$ and $\alpha \wedge (d\alpha)^n > 0$

Example

(a) $(\mathbb{R}^3, \xi_{std} = \ker \alpha_{std})$, where $\alpha_{std} = dz - ydx$
(b) $(\mathbb{R}^3, \xi_{std} = \ker \alpha_{ot})$ with cylindrical coordinates $(r, \phi, z)$, where $\alpha_{ot} = \cos rdz + r \sin rd\phi$
Figure: Tight vs. OT
Tight vs. OT

Definition

(1) Overtwisted (OT) disk is an embedded disk $D \subset (M, \xi)$ such that $\xi_p = T_pD$, $\forall p \in \partial D$ and characteristic foliation $D_\xi$ contains a unique singular point in the interior of $D$.

(2) If $(M, \xi)$ has an OT disk, then we call $(M, \xi)$ OT contact manifold and otherwise, tight contact manifold.

Figure: Overtwisted disk
Motivation

Theorem (Honda-Kazez-Matic, 2003)
Let $M = \Sigma \times I/ \sim$ whose fiber $\Sigma$ is genus $g > 1$, closed orientable surface and the monodromy map is pseudo-Anosov. Assume that $|\langle e(\xi), \Sigma \rangle| = |\chi(\Sigma)|$ holds for each fiber $\Sigma$. Then there exists a unique tight contact structure up to isotopy in each case.

Theorem (Colin-Giroux-Honda, 2009)
Every closed, atoroidal 3-manifold admits at most finitely many tight contact structures up to contact isotopy.

Conjecture
Every closed hyperbolic 3-manifold admits tight contact structures.
Convex surfaces

Definition

(1) A contact vector field \( \nu \) on \((M^{2n+1}, \xi)\) is a vector field with the property \((\psi_t)_*(\xi) = \xi, \ t \in \mathbb{R}\) where \(\psi_t\) is a local flow of \(\nu\).

(2) A convex surface \(\Sigma\) in \((M^3, \xi)\) is an embedded surface with contact vector field \(\nu\) near \(\Sigma\) s.t. \(\nu \pitchfork \Sigma\).

Theorem

Every closed, orientable surface \(\Sigma\) in \((M^3, \xi)\) has a convex surface which is isotopic and \(C^\infty\)-close to \(\Sigma\).
Dividing curves

Definition
The dividing set $\Gamma_\Sigma$ of a convex surface $\Sigma \subset (M, \xi)$ (w.r.t. a given contact vector field $v$) is the set of points $p \in \Sigma$ s.t. $v_p \in \xi_p$.

Proposition (Properties of dividing curves)

(1) $\Gamma$ : disjoint union of closed curves (nonempty)
(2) $\Gamma \pitchfork \Sigma_\xi$
(3) Isotopy class of $\Gamma_\Sigma$ does not depend on the choice of $v$
(4) $\Sigma \setminus \Gamma_\Sigma = R_+ (\Gamma_\Sigma) \sqcup R_- (\Gamma_\Sigma)$
(5) Convex surface $\Sigma \subset (M, \xi)$ has a tight neighborhood,

$$\iff \Gamma_\Sigma = \begin{cases} 
\text{single curve homotopic to } S^1, & \text{if } \Sigma = S^2 \\
\text{homotopically trivial curve}, & \text{otherwise.}
\end{cases}$$
Figure: Example of dividing set
Convex decomposition theory

\[ M \]

\[ S \]

\[ \text{cut along } S \]

\[ \text{glue along } S \] (Colin)

\[ B^3 \]

(Elashberg)
Giroux’s Flexibility Theorem

Theorem
Assume $\Sigma$ is convex with characteristic foliation $\Sigma_\xi$, contact vector field $v$ and dividing set $\Gamma_\Sigma$. Let $\mathcal{F}$ be another singular foliation on $\Sigma$ which is adapted to $\Gamma_\Sigma$ (i.e., there is a contact structure $\xi'$ in a neighborhood of $\Sigma$ such that $\Sigma_{\xi'} = \mathcal{F}$ and $\Gamma_\Sigma$ is also a dividing set for $\xi'$). Then there is an isotopy $\phi_s$, $s \in [0, 1]$, of $\Sigma$ in $(M, \xi)$ such that:

1. $\phi_0 = id$ and $\phi_s|_{\Gamma_\Sigma}$ for all $s$.
2. $\phi_s(\Sigma) \pitchfork v$ for all $s$.
3. $\phi_1(\Sigma)$ has characteristic foliation $\mathcal{F}$.

By Giroux’s flexibility theorem, we can think that the isotopy type of the dividing set of convex surface determines a contact structure of neighborhood of convex surface up to contact isotopy.
Legendrian Realization Property (LeRP)

Definition

1. A submanifold $L \subset (M^{2n+1}, \xi)$ is Legendrian if $\dim L = n$ and $T_xL \subset \xi_x$ for all $x \in L$.

2. The twisting number $t(L, F)$ is the integer difference in the number of twists between the normal framing and $F$, where $F$ is some fixed framing for Legendrian knot $L$ and normal framing is induced from $\xi$ by taking $v_p \in \xi_p$ so that $(v_p, \dot{L}(p))$ form an oriented basis for $\xi_p$.

Theorem (Legendrian Realization)

Let $C$ be a nonisolating graph on a convex surface $S$ and $\nu$ a contact vector field transverse to $S$. Then there exists an isotopy $\phi_s, s \in [0, 1]$ such that

1. $\phi_0 = id, \phi_s|_{\gamma_s} = id$.
2. $\phi_s(S) \cap (and \ hence \ \phi_s(S) \ are \ all \ convex)$,
Edge-rounding

Figure: Edge rounding
Bypasses

Definition

Bypass half-disk is a convex half-disk $D$ with a Legendrian boundary satisfying $D \cap \Sigma = \alpha$, $D \cap \Sigma$ and $tb(\partial D) = -1$.

Lemma (Bypass Attachment Lemma)

Let $D$ be a bypass for $\Sigma$. There is a neighborhood of $\Sigma \cup D$ in $M$ which is diffeomorphic to $\Sigma \times [0, 1]$ with $\Sigma_i = \Sigma \times i$, $i \in 0, 1$ convex, $\Sigma \times [0, \epsilon]$ is $I$-invariant and $\Gamma_{\Sigma_1}$ is obtained by bypass move of $\Gamma_{\Sigma_0}$.

Figure: The effect of attaching bypass
Properties of Bypasses

- Abstract bypass can be realized.
- Every bypass can be slid locally.
- We can find a real bypass using Imbalance Principle.
- Digging/Attaching bypass.

Lemma (Imbalance Principle)

Let $S^1 \times [0, 1]$ be convex with Legendrian boundary inside a tight contact manifold. If $t(S^1 \times \{0\}) < t(S^1 \times \{1\}) \leq 0$, then there exists a bypass along $S^1 \times \{0\}$.

![Figure: (a) Abstract bypass move (b) Imbalance Principle](image_url)
Gluing

Theorem (Colin)

Let \((M, \xi)\) be an oriented, compact, connected, irreducible, contact 3-manifold and \(\Sigma \subset M\) be an incompressible convex surface with nonempty Legendrian boundary and \(\partial\)-parallel dividing set \(\Gamma_\Sigma\). If \((M \setminus \Sigma, \xi|_{M \setminus \Sigma})\) is universally tight, then \((M, \xi)\) is universally tight.

Theorem (Eliashberg’s Uniqueness Theorem)

If \(\xi\) is a contact structure in a neighborhood of \(\partial B^3\) that makes \(\partial B^3\) convex and the dividing set on \(\partial B^3\) consists of a single closed curve, then there is a unique extension of \(\xi\) to a tight contact structure on \(B^3\) (up to isotopy that fixes the boundary).
Existence of $\#\Gamma$-minimizing fiber.

Observation (L. 2014)

Let $M$ be a closed surface bundle over the circle with a closed oriented genus $g > 1$ surface fiber. If monodromy map is pseudo-Anosov, then there exists a convex surface which is isotopic to the fiber and $\#\Gamma$-minimizing, i.e., $\#\Gamma_\Sigma = 1$ or 2.

Proof.

- Cut $M$ along the convex fiber $\Sigma$. Since monodromy map is pseudo-Anosov, $\Gamma_{\Sigma_0} \neq \Gamma_{\Sigma_1}$.
- If $\Gamma_{\Sigma_0} \neq \Gamma_{\Sigma_1}$, then there exists an essential closed curve $\gamma$ such that $|\gamma \cap \Gamma_{\Sigma_0}| \neq |\gamma \cap \Gamma_{\Sigma_1}|$.
- Cut again $\Sigma \times [0, 1]$ along the convex annulus $\gamma \times I$. Then there are 3 types of bypass attachment arcs.
Type A: Done
Type B $\Rightarrow$ Type A
Type C $\Rightarrow$ Type B or C.
Proposition A (L. 2013)

Let $M = \Sigma \times I$ be a surface bundle over the interval, $\gamma_0$ be an arbitrary separating closed curve in $\Sigma \times \{0\}$ which is not homotopically trivial and $\epsilon$ be an arbitrary nonseparating closed curve intersecting $\gamma_0$ at two points. Take a separating closed curve $\gamma_1 = \tau_n^\epsilon \circ \gamma_0$ in $\Sigma \times \{1\}$ where $n \in \mathbb{Z}$. Fix dividing sets $\Gamma_{\Sigma_i} = \gamma_i$, $i = 0, 1$. Then the following inequalities hold.

(1) If $n > 1$, then

$$\#\pi_0(Tight(M)) \leq \begin{cases} 13 \cdot 3^{n-2}, & \text{if } g(\Sigma_+) > 1, \ g(\Sigma_-) > 1 \\ 14 \cdot 3^{n-2}, & \text{if } g(\Sigma_\pm) > 1, \ g(\Sigma_\mp) = 1 \\ 14, & \text{if } g = 2, \ n = 2 \\ 43 \cdot 3^{n-3} + \left[\frac{n}{2}\right], & \text{if } g = 2, \ n \geq 3 \end{cases}$$
If $n = 1$, then $\#\pi_0(\text{Tight}(M)) \leq 4$

If $n = 0$, then $\#\pi_0(\text{Tight}(M)) = 2$

If $n = -1$, then $\#\pi_0(\text{Tight}(M)) \leq 3$

If $n < -1$, then

$$\#\pi_0(\text{Tight}(M)) \leq \begin{cases} 
20, & \text{if } g(\Sigma_+) \neq 1 \text{ and } g(\Sigma_-) \neq 1 \\
24, & \text{otherwise}
\end{cases}$$

The lower bound is 2 for all cases.

**Remark**

The lower bound can be achieved using sutured Floer homology.
Figure: Base case
Proof of the upper bound of Proposition A

- We can think that Dehn twists occur inside a small annulus neighborhood of curve $\epsilon$.
- Cut $\Sigma_g \times I$ with a convex annulus $A = \epsilon \times I$.
- All possible configurations of dividing curves of annulus $A^+$ are the following. See figure.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figures.png}
\caption{All possibilities of dividing curves on $A_+$}
\end{figure}
Strategy

Step 1

\[ I_{k \in \mathbb{Z}_<0} \Rightarrow I_{k-1 \in \mathbb{Z}_\leq 0} \]
\[ I_{k \in \mathbb{Z}_{>1}} \Rightarrow I_{k+1 \in \mathbb{Z}_{>0}} \]
\[ \|_{2k}^{\pm} \Rightarrow \|_{2k-2}^{\pm} \]
\[ \|_{2k+1}^{\pm} \Rightarrow \|_{2k-1}^{\pm}, \text{ where } k \in \mathbb{Z}. \]
\[ \|_{1}^{\pm} \Rightarrow I_0 \text{ or } I_1. \]

Step 2

\( I_0 \) cannot admit a tight contact structure.

Step 3

\( I_0 \) admits at most 3 tight contact structures.

Step 4

\( I_1 \) case admits at most 1 tight contact structures which is different from \( I_0 \) type.
Step 1: \( I_{k \in \mathbb{Z}_{<0}} \to I_{k-1 \in \mathbb{Z}_{\leq 0}} \)
Step 3: $l_0$ admits at most 3 tight contact structures (1)
Step 3: $I_0$ admits at most 3 tight contact structures (2)
Step 3: $l_0$ admits at most 3 tight contact structures (3)

Figure: State transitions from $l_0$ by attaching bypasses
Proposition B (L. 2014)

Let $M$ be a surface bundle over the interval. Choose two arbitrary nonseparating curves $\gamma_0^1, \gamma_0^2$ in $\Sigma \times \{0\}$ which are not homotopically trivial and separate $\Sigma$ into two components. Let $\gamma_i^i$ be the image of $\gamma_i^0$ by pseudo-Anosov map for $i = 1, 2$. Fix dividing sets $\Gamma_{\Sigma_i} = \gamma_i^1 \cup \gamma_i^2$, $i = 0, 1$. Then

$$|?| \leq \#\pi_0(\text{Tight}(M)) \leq 9^d$$

where $d$ is a distance between $\Gamma_{\Sigma_0}$ and $\Gamma_{\Sigma_1}$ in the curve complex.
Theorem (L. 2014)

Let $M$ be a surface bundle over the circle with a closed oriented genus $g > 1$ surface fiber and pseudo-Anosov monodromy. Then there exist at least $\infty$ tight contact structures up to contact isotopy.
Thank you!