COBORDISMS OF LEFSCHETZ FIBRATIONS ON 4-MANIFOLDS

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(Generalized) Lefschetz fibrations

\[ f : V^{m+2k} \rightarrow M^{m+2} \]
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\[ f : V^{m+2k} \to M^{m+2} \]

s.t. at critical points is locally equivalent to the map

\[ \mathbb{R}^m \times \mathbb{C}^k \to \mathbb{R}^m \times \mathbb{C} \]

\[ (x, z_1, \ldots, z_k) \mapsto (x, z_1^2 + \cdots + z_k^2) \]
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The monodromy of a meridian of \( \text{Crit}(f) \) is a Dehn twist about a curve \( c \subset F_g \).

We have the monodromy representation

\[ \omega_f : \pi_1(M - \text{Crit}(f)) \to \text{Mod}_g. \]
Pullbacks

\[
\begin{array}{c}
V \\
\downarrow f \\
M
\end{array}
\]
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\[
\begin{array}{ccc}
V & \rightarrow & f \\
\downarrow & & \downarrow \\
N & \rightarrow & M
\end{array}
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\[ \left\{ \begin{array}{c}
\tilde{V} \xrightarrow{\tilde{q}} V \\
N \xrightarrow{q} M
\end{array} \right. \]

\text{Pullback of } f

\[ q^*(f) \]

\[ f \]
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Fiber preserving

Pullback of \( f \)
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Pullback of $f$
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Fiber preserving

$f$-regular
iff $q$ and $q|_{\partial N}$ transverse to $f$
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- Pullback of \( f \)
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- \( q^*(f) \)
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\( q: N \to M \) is \( f \)-regular iff \( q \) and \( q|_{\partial N} \) are transverse to \( f \).
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$q: N \to M$ is $f$-regular iff $q$ and $q|_{\partial N}$ are transverse to $f$.

\[
\tilde{V} = \{(x, v) \in N \times V \mid q(x) = f(v)\}
\]

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(q^*(f))(x, v) = x
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$q^*(f)$ has the same fiber of $f$. 
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In general we restrict to those \( f \) that belong to a given class of Lefschetz fibrations with fiber \( F: f \in \mathcal{L}(F) \). In this case we talk about \( \mathcal{L}(F) \)-universality.
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In general we restrict to those \( f \) that belong to a given class of Lefschetz fibrations with fiber \( F: f \in \mathcal{L}(F) \). In this case we talk about \( \mathcal{L}(F) \)-universality. In the following theorem, we refer to \( \mathcal{L}(F) \) as the class of Lefschetz fibrations over 2 or 3-manifolds.
There exist \( u_2 \) and \( u_3 \) that are universal for genus-\( g \) Lefschetz fibrations over 2- and 3-manifolds respectively. Moreover, universal Lefschetz fibrations can be characterized in terms of monodromy.

One of the main features of such \( u_i \) is that the monodromy representation \( \phi: u_i \mapsto \phi u_i \) is an isomorphism with \( \phi \) being the mapping class group of genus \( g \). Moreover,
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2-handles correspond to relators of a presentation of $\text{Mod}_g$ (having Dehn twists as generators).
Construction of $u_2 : U_2^6 \rightarrow M_2^4$
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Assume for simplicity \( g > 1 \).

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$v$ is universal for Lefschetz fibrations over surfaces with boundary (Z. 2012).
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The relators \( r_i \)'s are words in the \( \delta_i \)'s, so they can be represented by pairwise disjoint embedded loops

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is surjective. We kill the kernel by adding 2-handles \( H_i^2 \) to \( B^4 \) along \( r_i \) with an arbitrary framing (for example with framing 0).
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is surjective. We kill the kernel by adding 2-handles \( H^2_i \) to \( B^4 \) along \( r_i \) with an arbitrary framing (for example with framing 0). Let \( M_2 \) be the resulting 4-manifold:

\[ r_i \in \ker(\omega_{v'}) \implies v' \text{ extends over } H^2_i \sim \sim u_2. \]
Lefschetz cobordism groups

Fix the fiber genus $g$ and the dimension $m$ of the base manifolds.
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f_1: V_1 \to M_1 \quad \text{and} \quad f_2: V_2 \to M_2
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are cobordant iff

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Let $U_g$ be the set of equivalence classes.

$f_1: V_1 \to M_1$ and $f_2: V_2 \to M_2$

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for a Lefschetz fibration $h: W \to N$.

Let $\Lambda(g, m)$ be the set of equivalence classes.

It’s an abelian group called the Lefschetz cobordism group.
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\( \Omega_n(M) \) is the n-th bordism group of M (a topological invariant), whose elements are bordism classes of maps

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with \(N\) a closed oriented \(n\)-manifold.

\(q\) is bordant to \(q' : N' \to M\) \iff \exists a simultaneous extension \(Q : W \to M\) with \(\partial W = N \sqcup (-N')\).
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**Corollary.** $u: U \to M$ universal with respect to Lefschetz fibrations over $n$-manifolds implies $u_*: \Omega_n(M) \to \Lambda(g, n)$ surjective. So, $u_{2*}: \Omega_2(M_2) \to \Lambda(g, 2)$, $u_{3*}: \Omega_2(M_3) \to \Lambda(g, 2)$, and $u_{3*}: \Omega_3(M_3) \to \Lambda(g, 3)$ are surjective.
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Under certain conditions, the natural homomorphism

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hence $\Lambda(g, 2)$ is a finitely generated abelian group ($k = \#\{\text{relators}\}$).

**Consequence:** $\Lambda(g, 2)$ is generated by the relators in a presentation of $\text{Mod}_g$. 
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We have

$$H_2(\text{Mod}_g) \cong \mathbb{Z} \quad (g \geq 4)$$

(Harer 1982 (incorrect) and 1985 for $g \geq 5$, later Korkmaz & Stipsicz 2003 for $g \geq 4$, based on work of Pitsch 1999).
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(4) How to construct Lefschetz fibrations that are universal with respect to higher dimensional base manifolds?
Questions

(1) How can we compute $\Lambda(g,m)$?

(2) In light of $H_2(\text{Mod}_g) \subset \Lambda(g, 2)$, $g \geq 3$, may we consider $\Lambda(g, 2)$ as an “enhanced” second homology of $\text{Mod}_g$?

(3) What properties of $\text{Mod}_g$ reflect to $\Lambda(g, 2)$?

(4) How to construct Lefschetz fibrations that are universal with respect to higher dimensional base manifolds?

(5) Is there a completion of the universal bundle $E\text{Mod}_g \to B\text{Mod}_g$ which is a universal Lefschetz fibration (for all base manifolds)?
Thank you for your attention!!