On the sum of tangle diagrams with certain symmetry

Byeorhi Kim

Department of Mathematics
College of Natural Sciences
Kyungpook National University

Aug 25. 2014
1. REVIEWS

2. CONSTRUCTION
   - Deformation of tangle diagram
   - Sum of two deformed tangle diagrams
   - Link diagram as the sum of two deformed tangle diagrams with certain symmetry

3. SOME PROPERTIES OF LINK DIAGRAM WITH CERTAIN SYMMETRY
Definitions

A $2n$-tangle is a pair of the unit 3-ball $B$ in $\mathbb{R}^3$ and $n$-arcs properly embedded in $B$ so that the endpoints of $n$-arcs lie on the unit circle in the $yz$-plane.

A $2n$-tangle diagram is the projection of $2n$-tangle onto $yz$-plane added over/under information to each of the double point. In this talk, we consider $2n$-tangle diagrams with $2n$-endpoints.
**Definition**
A $2n$-tangle diagram is said to be *descending* if we can assign suitable height to each arc so that the crossing between arcs are compatible with the height.

**Definition**
A $2n$-tangle diagram is said to be *alternating* if the crossings in the tangle alternate from under to over as we go along any arc of the weave.
Definition
For given two $2n$-tangle diagrams $T_1$ and $T_2$, we can get a link diagram $T_1 \boxplus T_2$ on the plane in $\mathbb{R}^2$ or 2-sphere $S^2$, called a sum of $T_1$ and $T_2$, by connecting them naturally.
Let $T$ be a $2n$-tangle diagram. Fix a point $P$ on the circle of $T$ except $2n$-endpoints. A deformed tangle diagram $(T, P)$ is a diagram obtained by cutting the circle of $T$ at the point $P$. We get two deformed tangle diagrams from $T$ by choosing a direction of circle.
Sum of two deformed tangle diagrams

Let \((T_1, P_1)\) and \((T_2, P_2)\) be two deformed \(2n\)-tangle diagrams. A sum of two deformed tangle diagrams is defined by connecting each endpoint of \((T_1, P_1, -)\) with the endpoint in the same position of \((T_2, P_2, +)\) in parallel as described in Fig. Then it is well defined. We denote \((T_1, P_1) \boxplus (T_2, P_2)\).
Sum of two deformed tangle diagrams

Let \((T_1, P_1)\) and \((T_2, P_2)\) be two deformed \(2n\)-tangle diagrams. A sum of two deformed tangle diagrams is defined by connecting each endpoint of \((T_1, P_1, -)\) with the endpoint in the same position of \((T_2, P_2, +)\) in parallel as described in Fig. Then it is well defined. We denote \((T_1, P_1) \boxplus (T_2, P_2)\).
Sum of two deformed tangle diagrams

Let \((T_1, P_1)\) and \((T_2, P_2)\) be two deformed \(2n\)-tangle diagrams. A sum of two deformed tangle diagrams is defined by connecting each endpoint of \((T_1, P_1, -)\) with the endpoint in the same position of \((T_2, P_2, +)\) in parallel as described in Fig. Then it is well defined. We denote \((T_1, P_1) \boxplus (T_2, P_2)\).
Sum of two deformed tangle diagrams

Let \((T_1, P_1)\) and \((T_2, P_2)\) be two deformed \(2n\)-tangle diagrams. A sum of two deformed tangle diagrams is defined by connecting each endpoint of \((T_1, P_1, -)\) with the endpoint in the same position of \((T_2, P_2, +)\) in parallel as described in Fig. Then it is well defined. We denote \((T_1, P_1) \boxplus (T_2, P_2)\).
Link diagram as the sum of two deformed tangle diagrams with certain symmetry

Let $T$ be a deformed $2n$-tangle diagram. Then naturally we get new deformed $2n$-tangle diagrams from $T$.

$T^*$ is obtained by changing all crossings of $T$.

$T^*_R$ is obtained by reflecting $T$ horizontally.

$T^*_R$ is obtained by changing all crossings of $T^*_R$.

The sum of a deformed tangle diagram $T$ with a derived tangle diagram from $T$ has certain symmetry. Now we will study about some properties of link diagrams with certain symmetry.
On the sum of tangle diagrams with certain symmetry
Note

Let \( (T, P, -) \) be a deformed \( 2n \)-tangle diagram.

1. \( (T, P, +) \) is the same tangle diagram as the rotating \( (T, P, -) \) through 180 degree.

2. \( (T_R, P, +) \) is the same tangle diagram as the vertical reflection of \( (T, P, -) \).
Some properties of link diagrams with certain symmetry

Theorem
Let \( T \) be a deformed \( 2n \)-tangle diagram. Then \( T \boxplus T_R \) and \( T \boxplus T^*_R \) are link diagrams with \( n \)-components.

Proof.
Let \( \alpha_k \) be an arc of \( T \) and \( e_i, e_j \) endpoints of \( \alpha_k \). Let \( (\alpha_R)_k \) be an arc of \( T_R \) corresponding to \( \alpha_k \) and \( (e_R)_i, (e_R)_j \) endpoints of \( (\alpha_R)_k \). Then \( e_i \) is connected with \( (e_R)_i \) and \( e_j \) is connected with \( (e_R)_j \). So we get a component by connecting \( \alpha_k \) and \( (\alpha_R)_k \). On the other hand, \( \alpha_k \) is not connected with \( (\alpha_R)_l \) for \( l \neq k \). Therefore \( T \boxplus T_R \) is a link diagram with \( n \)-components.
Example
Let $T$ be a deformed 6-tangle diagram. Then $T \boxplus T_R$ and $T \boxplus T_R^*$ are link diagrams with 3-components.
Lemma

If $T$ is a reduced alternating, then $T \boxplus T_R$ is a reduced alternating.

Sketch of proof.

Theorem

Let $T$ be a deformed $2n$-tangle diagram. Then $c(T \boxplus T_R) \leq 2c(T)$. Moreover, if $T$ is a reduced alternating, then $'\equiv'$ holds.
Theorem

Let $T$ be a deformed $2n$-tangle diagram and $L_1, \cdots, L_n$ $n$-components of $T \boxplus T_R^*$. Then $\text{lk}(L_k, L_l) = 0$ for $k \neq l$.

Sketch of proof.
Theorem
If the diagram of $T$ contains a triangle composed a crossing with two end points, then the crossing can be removed in $T \boxplus T^*_R$.

Proof.
Let $T$ be a deformed $2n$-tangle diagram. Let $c_k$ be a crossing of $T$ composed $\alpha_i$ and $\alpha_j$. Suppose that $\alpha_i$ is over and $\alpha_j$ is under. Then $(\alpha^*_R)_i$ and $(\alpha^*_R)_j$ compose a crossing of $T^*_R$, where $(\alpha^*_R)_i$ and $(\alpha^*_R)_j$ are arcs in $T^*_R$ corresponding to $\alpha_i$ and $\alpha_j$, respectively. Since $T^*_R$ is obtained by changing all crossings of $T_R$, $(\alpha^*_R)_i$ is over and $(\alpha^*_R)_j$ is under. So we get 2-bridges by connecting $\alpha_i \boxplus (\alpha^*_R)_i$ and $\alpha_j \boxplus (\alpha^*_R)_j$. By RM II, we can remove the crossing. $\square$
On the sum of tangle diagrams with certain symmetry

2 bridge

RM II
Example

Let $T$ be a deformed $2n$-tangle diagram. If $T$ is descending, then $T \boxplus T_R^*$ is the trivial link diagram with $n$-components.
Theorem
Let $T$ be a deformed $2n$-tangle diagram. Then $T \boxplus T$ is a 2-periodic link diagram.

Example

trefoil  figure-eight
Thank you for your attention.