Combinatorial torsions and the cobordism group of homology cylinders

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December 19, 2009

Overview

- Homology cylinder
- The homology cobordism group of homology cylinders
- Main Theorem
- The Torelli group anoalogue
- Outline of the proof

Homology cylinder

- For $g, k \geq 0$, let $\Sigma_{g,k} :=$ oriented compact surface of genus g with k boundary components.
- A homology cylinder (M, i_+, i_-) over $\Sigma_{g,k}$ is a 3-manifold M together with two embeddings $i_+, i_- : \Sigma_{g,k} \to \partial M$ such that
  1. (1) i_+ is orientation preserving and i_- is orientation reversing,
  2. (2) $\partial M = i_+(\Sigma_{g,k}) \cup i_-(\Sigma_{g,k})$ and $i_+(\Sigma_{g,k}) \cap i_-(\Sigma_{g,k}) = i_+(\partial\Sigma_{g,k}) = i_-(\partial\Sigma_{g,k})$,
  3. (3) $i_+|_{\partial\Sigma_{g,k}} = i_-|_{\partial\Sigma_{g,k}}$,
  4. (4) $i_+, i_- : H_*(\Sigma_{g,k}; \mathbb{Z}) \to H_*(M; \mathbb{Z})$ are isomorphisms.

Homology cylinder: Examples

- $\mathcal{M}_{g,k}$:= the mapping class group.
  For a diffeomorphism $\varphi \in \mathcal{M}_{g,k}$,
  $$M(\varphi) = (\Sigma_{g,k} \times [0,1]/\sim,\ i_+ = \mathrm{id} \times 0,\ i_- = \varphi \times 1).$$
  where $\sim$ is given by $(x,s) \sim (x,t)$ for $x \in \partial\Sigma_{g,k}$ and $s,t \in [0,1]$ is a homology cylinder.
- Let K be a knot of genus g such that $\Delta_K(t)$ is monic and such that $\deg(\Delta_K(t)) = 2g$. Let $\Sigma \subset S^3 - N(K)$ be a minimal genus Seifert surface. Then $(S^3 - N(K)) - \Sigma \times (0,1)$ is a homology cylinder over $\Sigma_{g,1}$ in a natural way.

Homology cylinder

- Two homology cylinders (M, i_+, i_-) and (N, j_+, j_-) over $\Sigma_{g,k}$ are called isomorphic if there exists an orientation preserving diffeomorphism $f : M \to N$ satisfying $j_+ = f \circ i_+$ and $j_- = f \circ i_-$.
- $\mathcal{C}_{g,k}$:= the monoid of all isomorphism classes of homology cylinders over $\Sigma_{g,k}$.
- The product operation on $\mathcal{C}_{g,k}$:

$$(M, i_+, i_-) \cdot (N, j_+, j_-) := (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-).$$

Homology cobordism of homology cylinders

- Two homology cylinders (M, i_+, i_-) and (N, j_+, j_-) over $\Sigma_{g,k}$ are called homology cobordant if there exists a compact oriented smooth 4-manifold W such that

$$\partial W = M \cup (-N) \,/\, (i_+(x) = j_+(x), i_-(x) = j_-(x), x \in \Sigma_{g,k}),$$

  and such that the inclusion induced maps $H_*(M;\mathbb{Z}) \to H_*(W;\mathbb{Z})$ and $H_*(N;\mathbb{Z}) \to H_*(W;\mathbb{Z})$ are isomorphisms.
- $\mathcal{H}_{g,k}$:= the group of homology cobordism classes of elements in $\mathcal{C}_{g,k}$

- For a diffeomorphism $\varphi \in \mathcal{M}_{g,k}$,

$$M(\varphi) = (\Sigma_{g,k} \times [0,1]/\sim,\ i_+ = \mathrm{id} \times 0,\ i_- = \varphi \times 1).$$

where $\sim$ is given by $(x,s) \sim (x,t)$ for $x \in \partial\Sigma_{g,k}$ and $s,t \in [0,1]$ is a homology cylinder.

Theorem (Garoufalidis-Levine)

$\mathcal{M}_{g,k}$ embeds into $\mathcal{C}_{g,k}$ and $\mathcal{H}_{g,k}$.

Homology cylinder

- If $g \geq 3$, then $\mathcal{M}_{g,k}$ is perfect: $\mathcal{M}_{g,k} = [\mathcal{M}_{g,k}, \mathcal{M}_{g,k}]$.

Theorem (2009, Goda-Sakasai)

For $g \geq 1$, $\mathcal{C}_{g,1}$ surjects to $\mathbb{Z}^\infty$, hence not perfect.

- Idea of proof: rank of sutured Floer homology $SFH(M, i_+(\partial\Sigma_{g,1}))$.

Question

Questions (Garoufalidis-Levine, Goda-Sakasai)

Is $\mathcal{H}_{g,k}$ infinitely generated? Is $\mathcal{H}_{g,k}$ not perfect?

- $\mathcal{H}_{0,0}^{\text{smooth}} \cong \mathcal{H}_{0,1}^{\text{smooth}} \cong \Theta_3^{\text{smooth}}$ and it has infinite rank.
- $\mathcal{H}_{0,0}^{\text{top}} \cong \mathcal{H}_{0,1}^{\text{top}} \cong \Theta_3^{\text{top}} = 0$
- $\mathcal{H}_{0,2} \cong \mathbb{Z} \oplus \mathit{Conc} \twoheadrightarrow (\mathbb{Z}/2)^\infty \oplus (\mathbb{Z}/4)^\infty \oplus \mathbb{Z}^\infty$

Main Theorem

Theorem (Cha-Friedl-K.)

(1) *If* $b_1(\Sigma_{g,k}) > 0$*, then there exists an epimorphism*

$$\mathcal{H}_{g,k} \to (\mathbb{Z}/2)^\infty$$

which splits. In particular, the abelianization of $\mathcal{H}_{g,k}$ *contains a direct summand isomorphic to* $(\mathbb{Z}/2)^\infty$*.*

(2) *If* $k > 1$*, then there exists an epimorphism*

$$\mathcal{H}_{g,k} \to \mathbb{Z}^\infty.$$

Furthermore, the abelianization of $\mathcal{H}_{g,k}$ *contains a direct summand isomorphic to* $(\mathbb{Z}/2)^\infty \bigoplus \mathbb{Z}^\infty$*.*

Corollary

If $b_1(\Sigma_{g,k}) > 0$*, then* $\mathcal{H}_{g,k}$ *is infinitely generated and not perfect. Furthermore,* $\mathcal{H}_{g,k}$ *is not finitely related.*

The Torelli group analogue

- The Torelli group $\mathcal{I}_{g,k}$ of $\mathcal{M}_{g,k}$ is

$$\mathcal{I}_{g,k} := \{g \in \mathcal{M}_{g,k} \mid g \text{ acts trivially on } H_1(\Sigma_{g,k})\}.$$

- For a homology cylinder (M, i_+, i_-) over Σ, we can define the automorphism

$$\varphi(M) := (i_+)_*^{-1}(i_-)_* \colon H_1(\Sigma) \xrightarrow[(i_-)_*]{\cong} H_1(M) \xrightarrow[(i_+)_*^{-1}]{\cong} H_1(\Sigma).$$

Then $\varphi(M) \in \mathrm{Aut}^*(H) \subset \mathrm{Aut}(H)$ for $H = H_1(\Sigma)$.

- The Torelli group of $\mathcal{H}_{g,k}$ is

$$\mathcal{IH}_{g,k} := \{(M, i_+, i_-) \in \mathcal{H}_{g,k} \mid \varphi(M) = \mathrm{id}\}.$$

The Torelli group analogue

Theorem

(1) *The group $\mathcal{I}_{g,n}$ is torsion-free,*

(2) *the group $\mathcal{I}_{g,n}$ is finitely generated for $g \geq 3$ and $n = 0, 1$,*

(3) *the group $\mathcal{I}_{2,0}$ is a free group on infinitely many generators,*

(4) *if $g \geq 3$, then the abelianization of $\mathcal{I}_{g,1}$ is isomorphic to $\mathbb{Z}^a \oplus (\mathbb{Z}/2)^b$ for some $a, b \in \mathbb{N}$.*

It is an open question whether the Torelli group $\mathcal{I}_{g,1}$ is finitely related for $g \geq 3$.

THEOREM (MORITA)

The abelianization of $\mathcal{IH}_{g,1}$ has infinite rank.

The Torelli group analogue

Theorem (Cha-Friedl-K.)

(1) *If* $b_1(\Sigma_{g,k}) > 0$, *then there exists an epimorphism*

$$\mathcal{IH}_{g,k} \to (\mathbb{Z}/2)^\infty$$

which splits. In particular, the abelianization of $\mathcal{IH}_{g,k}$ *contains a direct summand isomorphic to* $(\mathbb{Z}/2)^\infty$.

(2) *If* $g > 1$ *or* $k > 1$, *then there exists an epimorphism*

$$\mathcal{IH}_{g,k} \to \mathbb{Z}^\infty.$$

Furthermore, the abelianization of $\mathcal{IH}_{g,k}$ *contains a direct summand isomorphic to* $(\mathbb{Z}/2)^\infty \bigoplus \mathbb{Z}^\infty$.

Corollary

If $b_1(\Sigma_{g,k}) > 0$, then $\mathcal{IH}_{g,k}$ is infinitely generated and not perfect. Furthermore, $\mathcal{IH}_{g,k}$ is not finitely related.

Strategy for proving Main Theorem

(1) Let $H := H_1(\Sigma_{g,k})$. Construct a homomorphism

$$\mathcal{H}_{g,k} \to Q(H)^\times / \sim$$

for some quotient group of $Q(H)^\times$, the field of fractions of $\mathbb{Z}[H]$, using the Reidemeister torsion.

(2) Show that $\mathcal{H}_{g,k}$ has (the desired) nontrivial image in the abelian group $Q(H)^\times / \sim$ under the homomorphism.

Reidemeister torsion

- $H := H_1(\Sigma_{g,k})$ and $Q(H) :=$ the quotient field of $\mathbb{Z}[H]$.
- For a homology cylinder (M, i_+, i_-), let $\Sigma_\pm = i_\pm(\Sigma) \subset M$.
- Considering $\pi_1(M) \to H_1(M) \stackrel{\cong}{\leftarrow} H_1(\Sigma_+) \stackrel{i_+}{\leftarrow} H$, the torsion of (M, i_+, i_-) is defined by

$$\tau(M) = \tau(M, \Sigma_+; Q(H)) := \tau(C_*(\widetilde{M}, \widetilde{i_+(\Sigma_{g,k})}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(M)]} Q(H)).$$

- For any homology cylinder M, we have $\tau(M) = \operatorname{ord} H_1(M, \Sigma_+; \mathbb{Z}[H])$.
- A function $\mathcal{C}_{g,k} \xrightarrow{\tau} Q(H)^\times / \pm H$.

Construction of $\mathcal{H}_{g,k} \to Q(H)^\times / \sim$

Theorem

Let $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$ be homology cylinders over Σ. Then

$$\tau(M \cdot N) \doteq \tau(M) \cdot \varphi(M)(\tau(N)).$$

Theorem

Let $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$ be homology cylinders over a surface Σ which are homology cobordant. Write $H = H_1(\Sigma)$. Then

$$\tau(M) \doteq \tau(N) \cdot q \cdot \bar{q} \in Q(H)^\times$$

for some $q \in Q(H)^\times$.

Construction of $\mathcal{H}_{g,k} \to Q(H)^\times / \sim$

- $A(H) := \{\pm h \cdot p^{-1} \cdot \varphi(p) \mid h \in H,\ p \in Q(H)^\times, \text{ and } \varphi \in \mathrm{Aut}^*(H)\}$.
- $N(H) := \{\pm h \cdot q \cdot \bar{q} \mid q \in Q(H)^\times,\ h \in H\}$.
- Denote $A(H)N(H)$ by AN.

Theorem

The torsion invariant gives rise to a group homomorphism

$$\tau : \mathcal{H}_{g,k} \to Q(H)^\times / AN,$$

where $H = H_1(\Sigma_{g,k})$.

- In fact, there is a homomorphism

$$\mathcal{H}_{g,k} \to \mathrm{Aut}^*(H) \ltimes Q(H)^\times / N(H).$$

Structure of $Q(H)^\times/AN$

- $Q(H)^{sym} = \{p \in Q(H)^\times \mid p = \bar{p} \text{ in } Q(H)^\times/A\}$.
- There is an exact sequence:

$$1 \to \frac{Q(H)^{sym}}{AN} \to \frac{Q(H)^\times}{AN} \to \frac{Q(H)^\times}{Q(H)^{sym}} \to 1$$

- For $p, q \in \mathbb{Z}[H]$, define $p \sim q$ if $p = \varphi(q)$ for some $\varphi \in Aut^*(H)$.
- p is called self-dual if $p \sim \bar{p}$.
- For $p \in Q(H)$ and an irreducible $\lambda \in \mathbb{Z}[H]$,
  $e_\lambda(p)$= the sum of exponents of distinct irreducible factors μ of p such that $\mu \sim \lambda$.

Structure of $Q(H)^\times/AN$

- For each self-dual irreducible $\lambda \in \mathbb{Z}[H]$, define

$$\Psi_\lambda \colon Q(H)^{sym}/AN \to \mathbb{Z}/2$$

  by $\Psi_\lambda(p \cdot AN) = e_\lambda(p) \pmod 2$.

- For each non-self-dual irreducible $\mu \in \mathbb{Z}[H]$, define

$$\Theta_\mu \colon Q(H)^\times/Q(H)^{sym} \to \mathbb{Z}$$

  by $\Theta_\mu(p \cdot Q(H)^{sym}) = e_\mu(p) - e_{\bar{\mu}}(p)$.

Structure of $Q(H)^\times/AN$

Theorem

$$
\begin{array}{llcll}
(1) & \bigoplus_{[\lambda]} \Psi_\lambda & : & Q(H)^{sym}/AN & \xrightarrow{\cong} \bigoplus_{[\lambda]} \mathbb{Z}/2, \\
(2) & \bigoplus_{\{[\mu],[\bar{\mu}]\}} \Theta_\mu & : & Q(H)^\times/Q(H)^{sym} & \xrightarrow{\cong} \bigoplus_{\{[\mu],[\bar{\mu}]\}} \mathbb{Z}.
\end{array}
$$

Symmetry of τ

Lemma (Symmetry)

Suppose Σ is a surface with at most one boundary components. Then for any homology cylinder M over Σ, we have $\tau(M) = \overline{\tau(M)}$ in $Q(H)^\times/AN$.

Remark

If Σ has more than one boundary component, $\tau(M)$ is in general asymmetric even modulo AN. Using this, we can prove that the image of $\tau \colon \mathcal{H}_{g,k} \to Q(H)^\times/AN$ contains a subgroup isomorphic to $\mathbb{Z}^\infty$.