Satellite twisted torus knots

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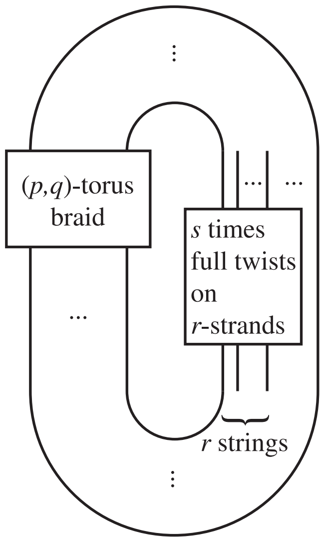
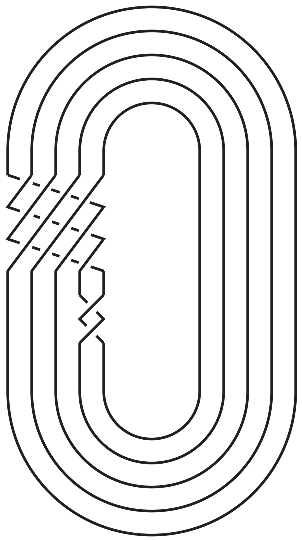
Twisted torus knots $T(p,q;r,s)$

p,q : coprime integers with $p>2$ and $q>1$
$T(p,q)$: the torus knot of type (p,q)
r : an integer with $p>r>1$
s : an integer $\neq 0$

T(*p*,*q*;*r*,*s*)



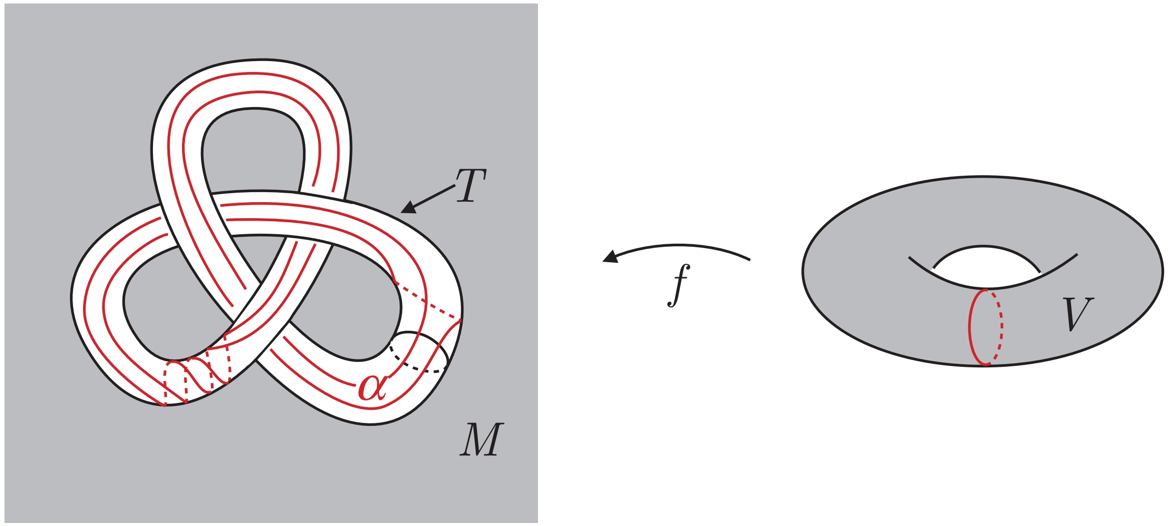


T(5,3;2,1)=*P*(-2,3,7)

Theorem . *Let* $T(p,q;r,s)$ *be a twisted torus knot with* $p>2, q>1$, *and* $p>r>1$.
(1) If r *is a multiple of* q, *then* $T(p,q;r,s)$ *is a cable of a torus knot.*
(2) If r *is not a multiple of* q, *then* $T(p,q;r,s)$ *is not a satellite knot when* $|s|\geq 3$.

Dehn fillings/surgeries

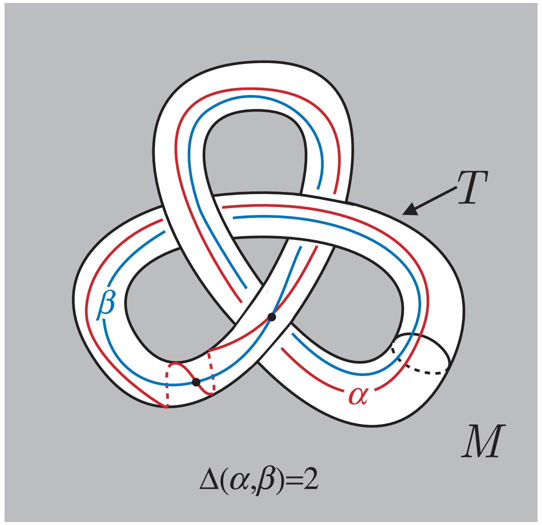
M is a compact, connected, orientable 3-manifold with a torus boundary component T.



$$M(\alpha) = M \cup_f V$$

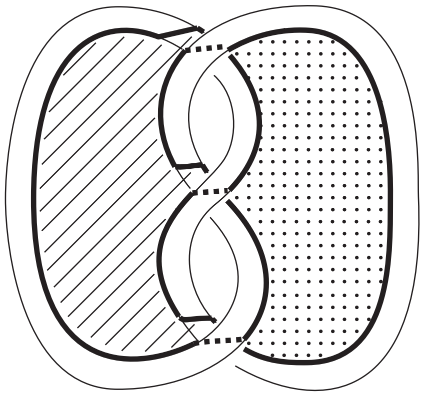
Slopes

The *slope* of an essential circle on T is its isotopy class ($T \subset \partial M$).
Let α, β be two slopes on T.
$\Delta(\alpha, \beta) :=$ minimal geometric intersection number of α and β.



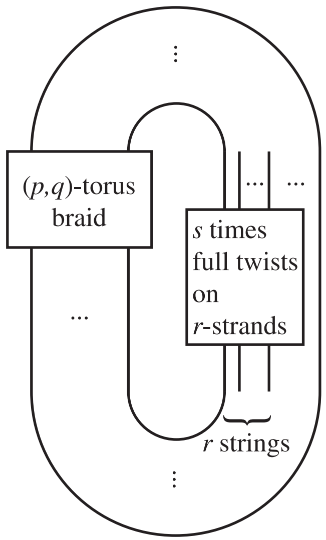
Parameterizing slopes

K : a knot in S^3, $E(K) = S^3 - \text{int}N(K)$
μ, λ : meridian and longitude $\subset \partial E(K)$
α : an essential simple closed curve in $\partial E(K)$
$\alpha \sim m\mu + l\lambda$ for some coprime integers m, l
$\{\text{slopes}\} \leftrightarrow \mathbb{Q} \cup \{1/0\}$
$\alpha \leftrightarrow m/l$



Twisted torus knots $T(p,q;r,s)$

$T(p,q;r,s)$ =



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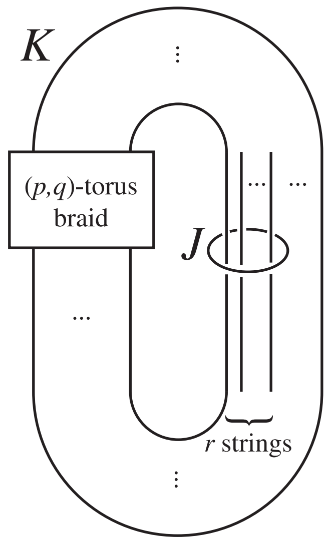
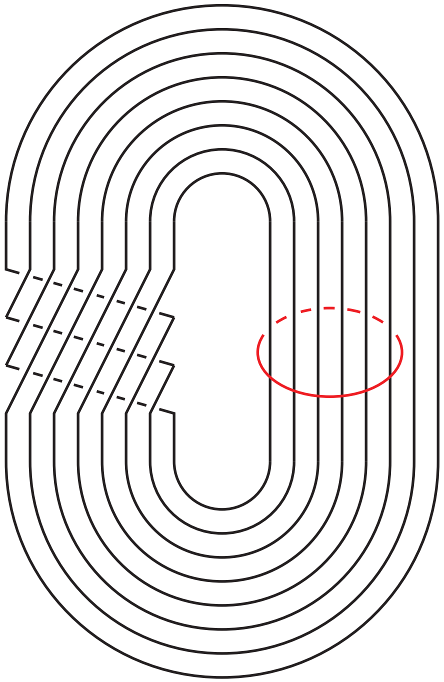
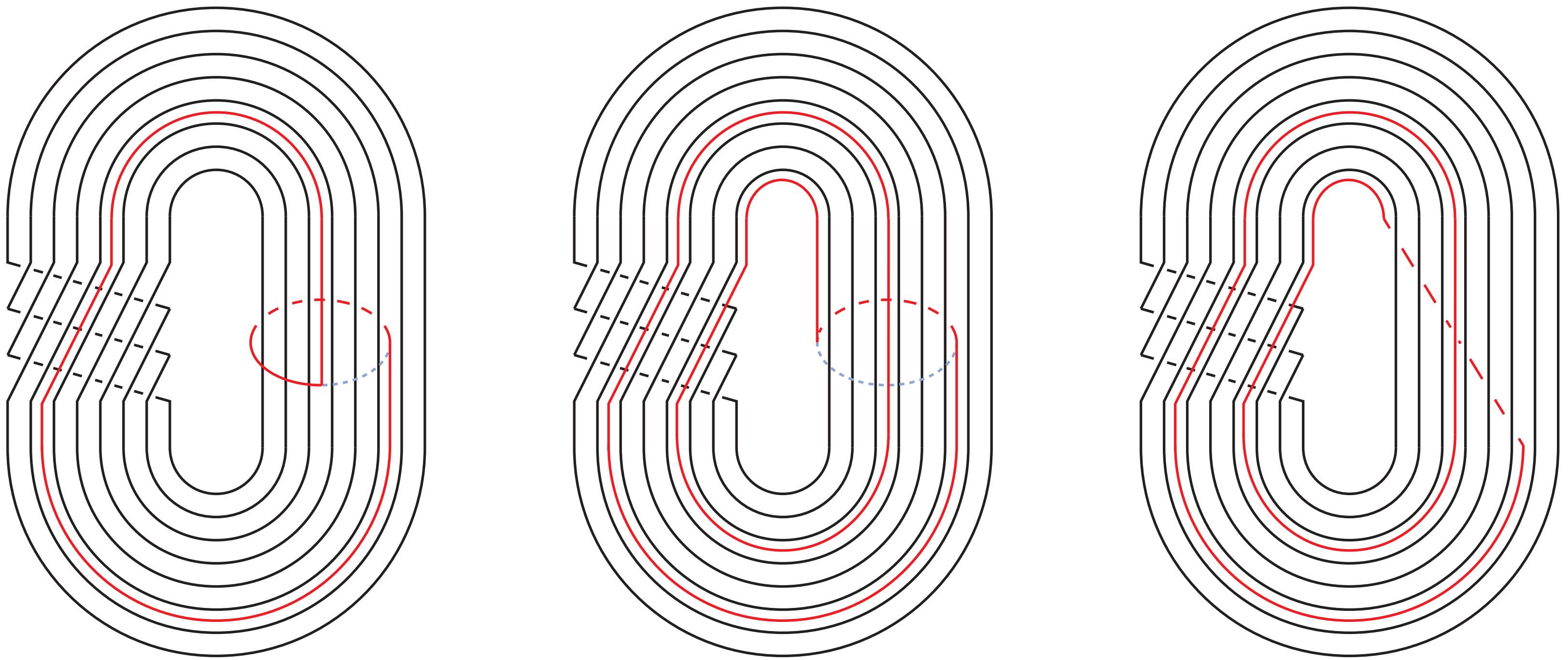


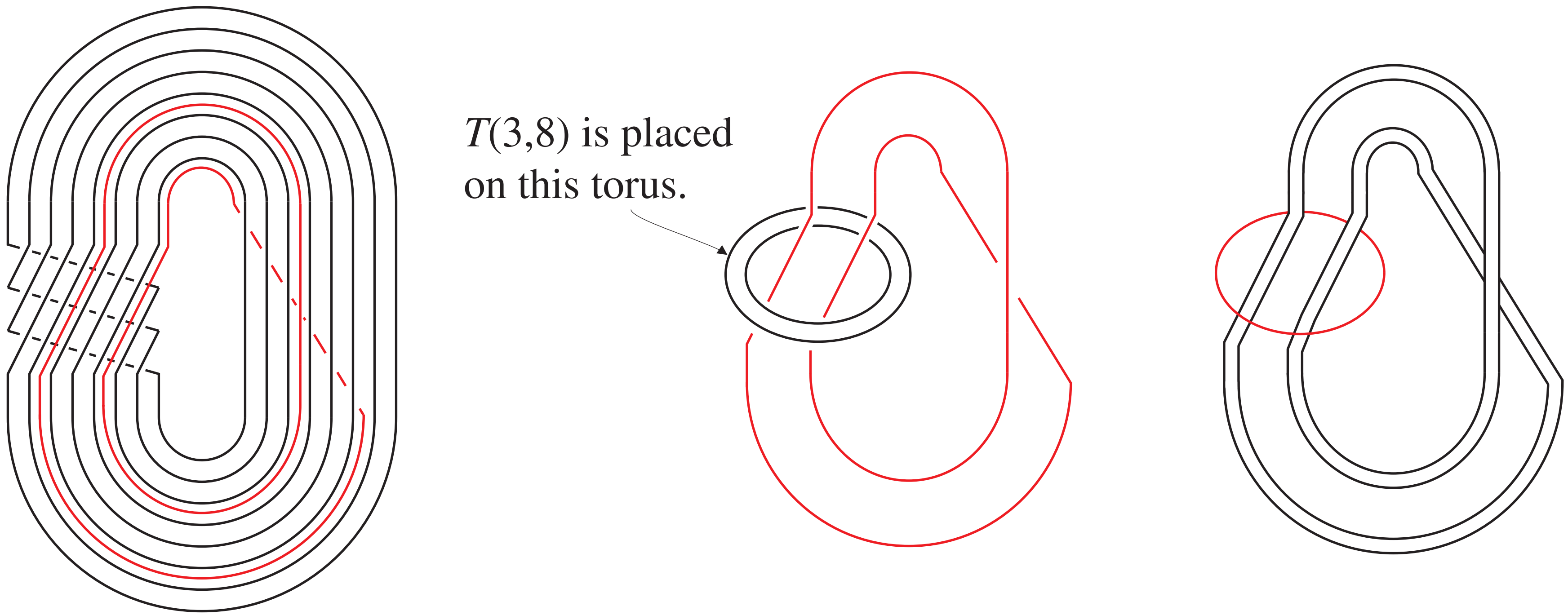
+ $(1/s)$-surgery on J

Assume that r is a multiple of q.

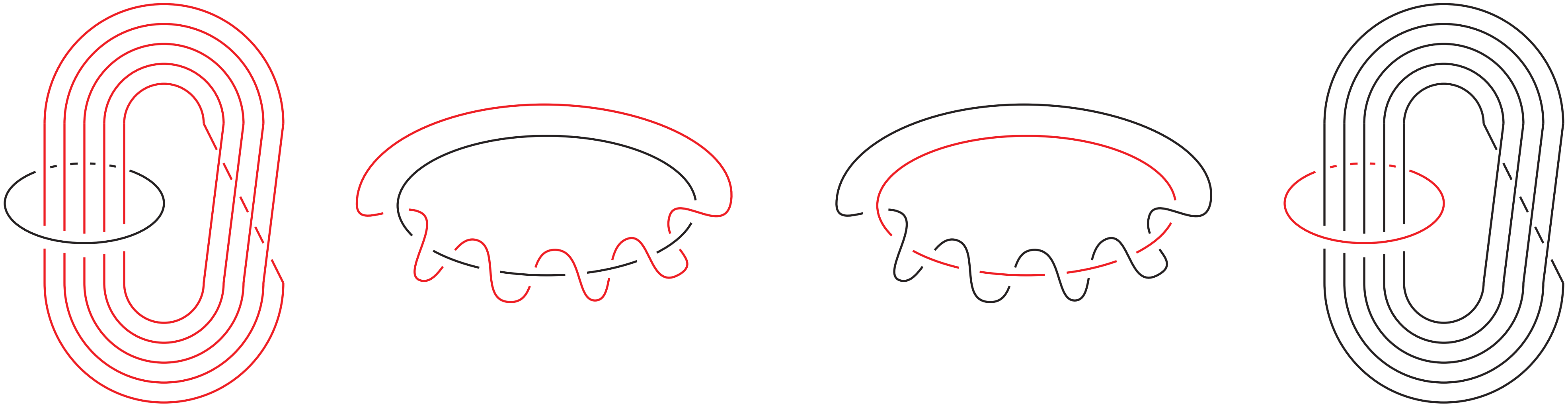


$T(8,3)$
$r=6$




$T(3,8)$ is placed
on this torus.

Assume $r = kq$.



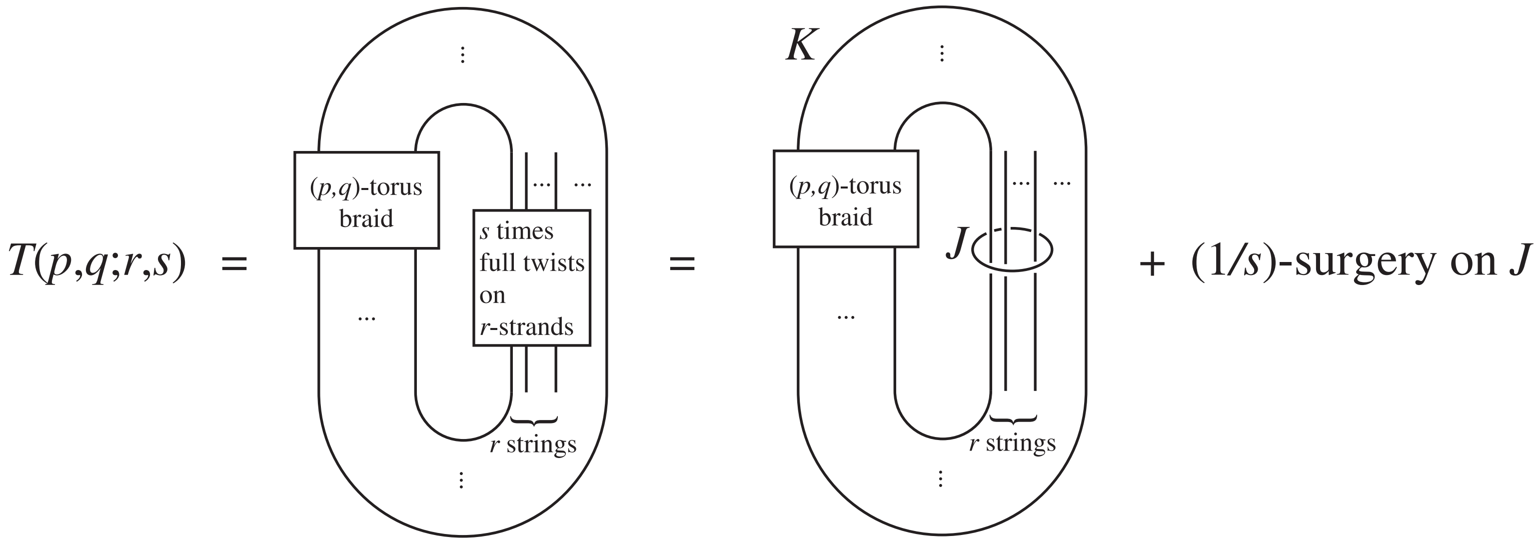
After $1/s$-surgery on J, $T(k,1)$ changes into $T(k,1+ks)$.

Theorem . *Assume* $r = kq$. *Then* $T(p,q;r,s)$ *is a cable of a torus knot* $T(k, 1+ks)$.

Remark . $T(k, 1+ks)$ *is the unknot if* $k = 1$ *or* $(k,s) = (2,-1)$. *In this case,* $T(p,q;r,s)$ *is a torus knot.*

Hereafter, we assume that r is not a multiple of q.

Let $M = S^3 - \text{int}N(J \cup K) = E(J \cup K)$.
Then M has two boundary tori, $\partial_J M$ and $\partial_K M$.



$M(1/0) = E(K) = E(T(p,q))$

$M(1/s) = E(T(p,q;r,s))$

$\Delta(1/0, 1/s) = |s|$

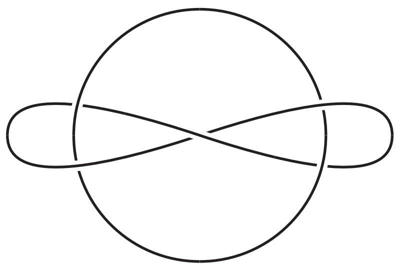
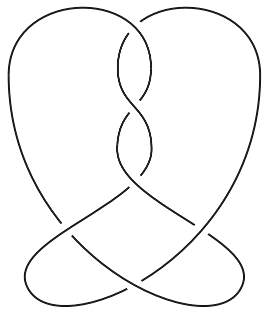
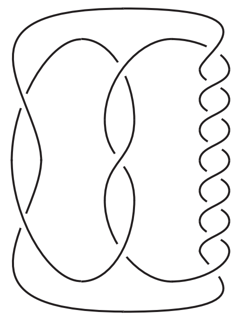
$M(1/0)$ contains an essential annulus.

Assume that $T(p,q;r,s)$ is a satellite knot. Then $M(1/s)$ contains an essential torus.

Definition . *A surface* F *in a 3-manifold is said to be* small *if* $F \cong S^2, D^2, A^2,$ *or* T^2*.*

Lemma . M *does not contain an essential small surface, that is,* M *is hyperbolic.*

Theorem (Gordon and Wu). *Let M be a hyperbolic 3-manifold such that $M(\alpha)$ contains an essential annulus and $M(\beta)$ contains an essential torus. Then $\Delta(\alpha, \beta) \leq 5$.*
Also, if $\Delta(\alpha, \beta) = 4$ or 5, then M is homeomorphic to one of the exteriors of the Whitehead link, 3/10 2-bridge link, the Whitehead sister link. Moreover, $M(\alpha)$ contains an essential torus.

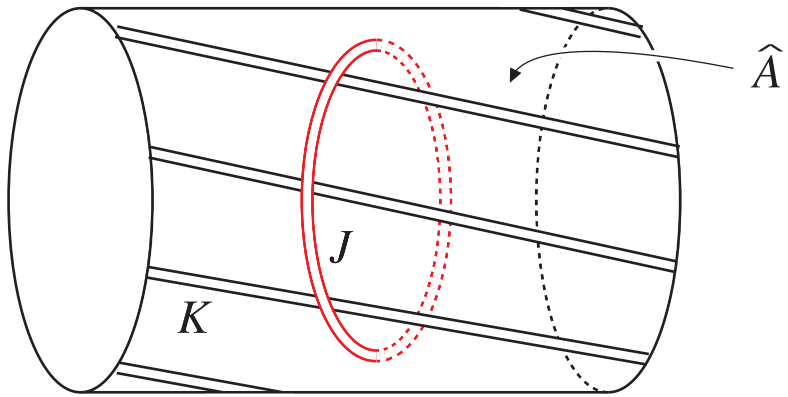
Lemma . $|s| \leq 3$.

Proof. By Gordon-Wu's result, $\Delta(1/0, 1/s) = |s| \leq 5$. If $|s| = 4$ or 5, then $M(1/0)$, which is the exterior of $T(p, q)$, must contain an essential torus. This is impossible. $\square$

From now on, we assume $\Delta(1/0, 1/s) = |s| = 3$.

$V_{1/0}, V_{1/s}$: attached solid tori in $M(1/0), M(1/s)$

$\widehat{A} \subset M(1/0)$: an essential annulus

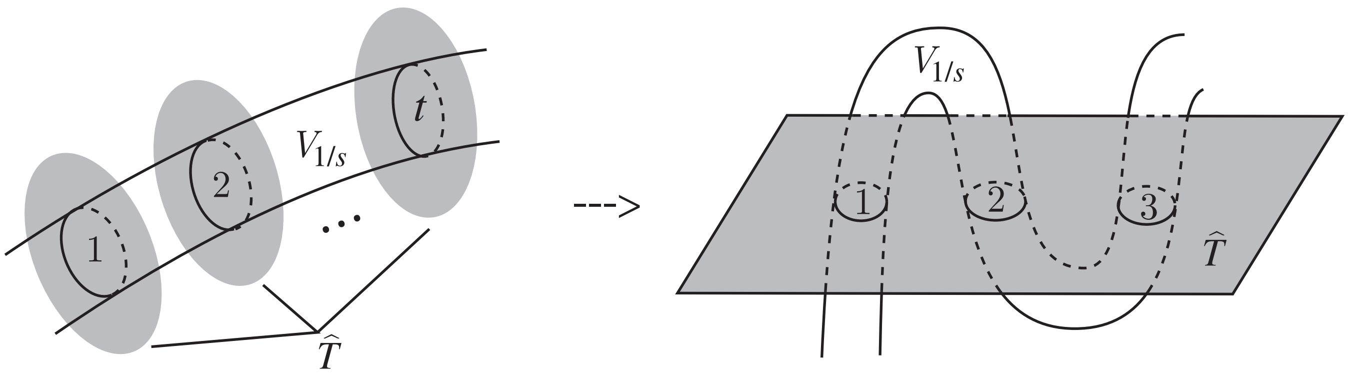


$\widehat{A} \cap V_{1/0} = u_1 \cup u_2$: meridian disks of $V_{1/0}$

$\hat{T} \subset M(1/s)$: an essential torus, which must be separating

$\hat{T} \cap V_{1/s} = v_1 \cup \ldots \cup v_t$: meridian disks of $V_{1/s}$

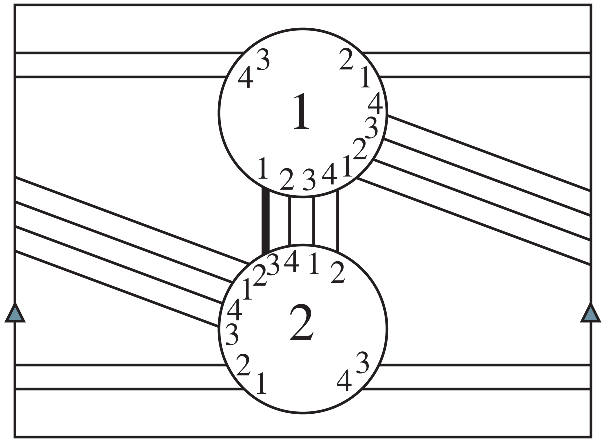
We assume that $\hat{T}$ had been chosen so that t is minimal.



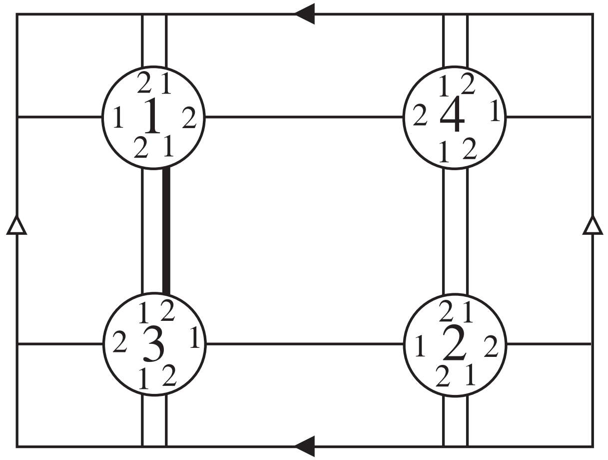
Let $A = \widehat{A} \cap M$ and $T = \widehat{T} \cap M$.
Then A and T are incompressible and ∂-incompressible.

Isotope A or T in M so that $A \pitchfork T$.

The arc components of $A \cap T$ define two labeled graphs G_A and G_T.
No trivial loop by ∂-incompressibility.

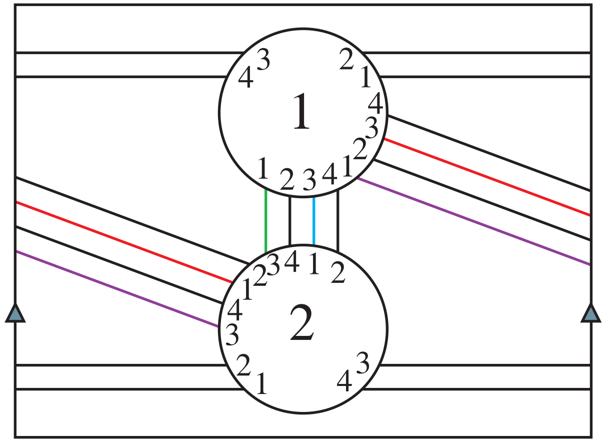


G_A

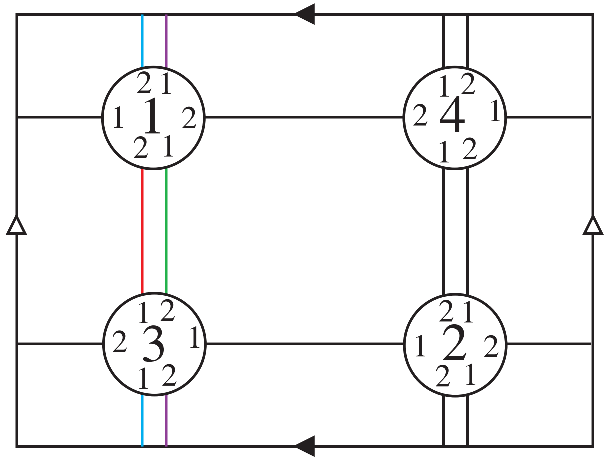


G_T

an example of the graph pair G_A, G_T

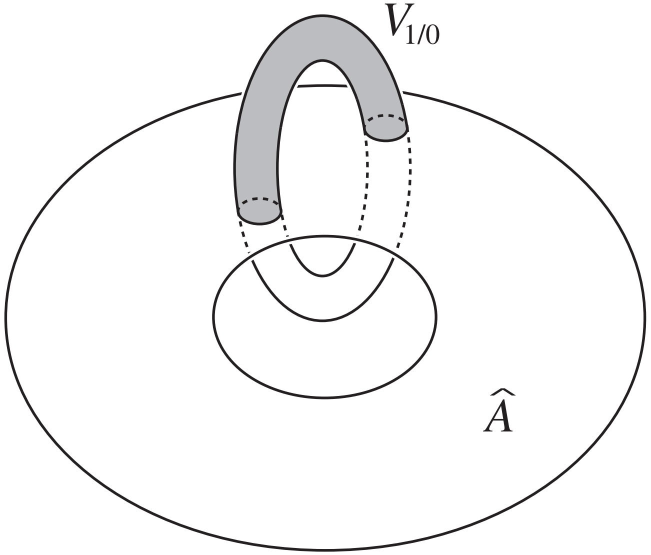


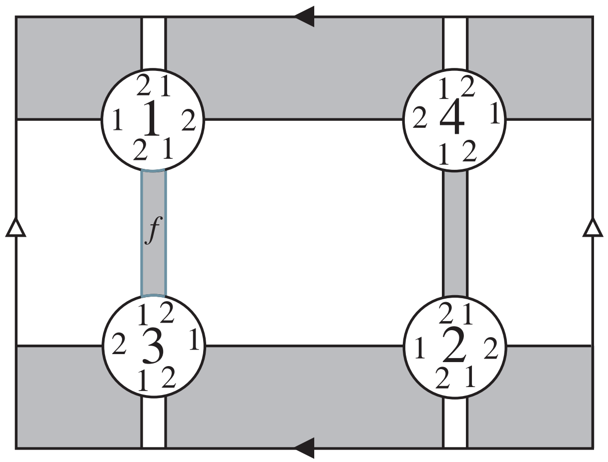
G_A



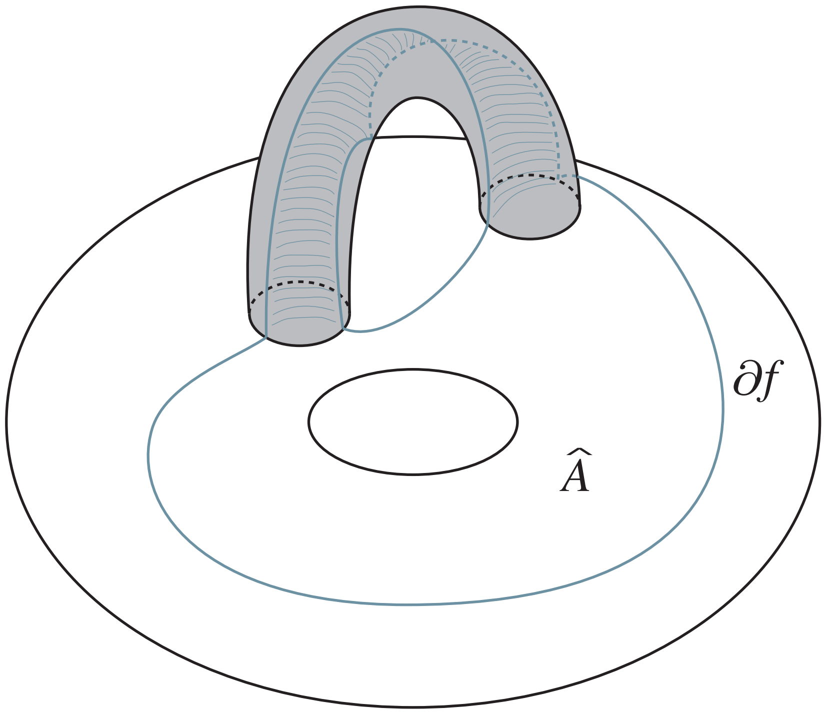
G_T

$\widehat{A} \subset M(1/0)$; $\widehat{A}$ divides $M(1/0)$ into two parts, black and white.





G_T


∂f
$\hat{A}$

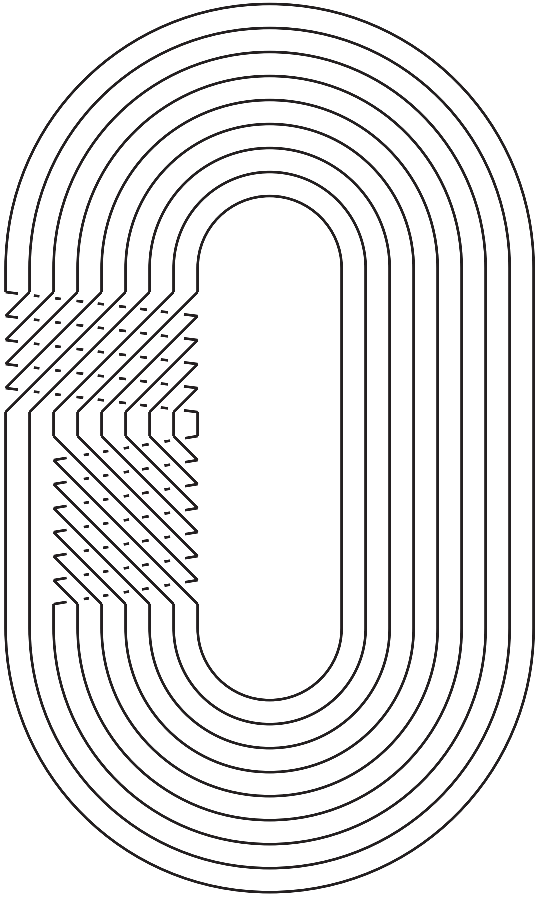
The core curve of $\widehat{A}$ bounds a Möbius band on each side of $\widehat{A}$.

B_B : a Möbius band on the black side of $\widehat{A}$
B_W : a Möbius band on the white side of $\widehat{A}$

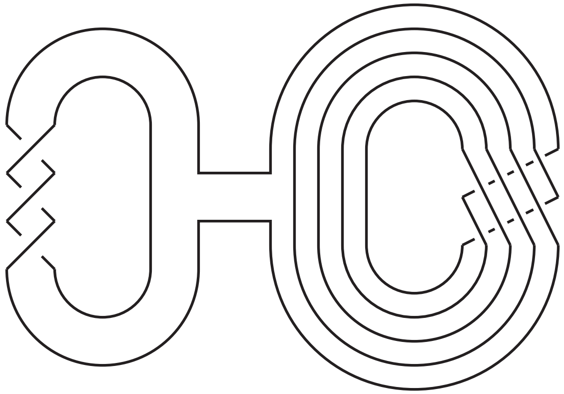
$B_B \cup B_W$ is a Klein bottle in $M(1/0) = E(K) \subset S^3$.

This is impossible.

Theorem . *Suppose that r is not multiple of q. Then $T(p,q;r,s)$ is not a satellite knot if $|s| \geq 3$.*



T(9,5;7,−1)



T(2,3) # *T*(5,−2)

Theorem (Morimoto). $T(p,q;r,s)$ *is a composite knot if and only if*

$$p = (a+1)k+1, q = ak+1, r = p-k_1, \text{ and } s = -1$$

for some $a > 0, k > 3, k_1 > 1$ *with* $k_1 < k-1$. *In this case,*

$$T(p,q;r,s) = T(k_1, ak_1+1)\#T((a+1)k_2+1, -k_2),$$

where $k_2 = k - k_1 > 1$.

Question . Are there any examples of $T(p, q; r, s)$ which are satellite knots with $|s| = 2$?