

Splittings of von Neumann rho–invariants

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von Neumann ρ -invariants

- ▶ M a closed 3-manifold, Γ a group
- ▶ Given $\phi : \pi_1(M) \rightarrow \Gamma$, whenever $(M, \phi) = \partial(W, \psi)$ for some compact oriented 4-manifold W and $\psi : \pi_1(W) \rightarrow \Gamma$

$$\rho(M, \phi) = \sigma_{\Gamma}^{(2)}(W, \phi) - \sigma_0(W) \in \mathbb{R}$$

where $\sigma_{\Gamma}^{(2)}$ is the von Neumann signature of the intersection form on $H_2(W; \mathbb{Z}\Gamma)$ and σ_0 is the ordinary signature.

- ▶ (Cheeger–Gromov 1985) ρ does not depend on W .
- ▶ (Cochran–Orr–Teichner 1999) If M_K is the zero surgery on a knot K in S^3 , $\Gamma = \mathbb{Z}$ and $\phi \neq 1$, then $\rho(M_K, \phi)$ is the integral of the Levine–Tristram signature function.
- ▶ A proper choice of Γ gives a concordance invariant.

Knot Concordance

A knot K in S^3 is *slice* if $K = \partial D$ for a locally flat 2-disk D in B^4 .

Two knots K_1 and K_2 are *concordant* if $K_1 \# -K_2$ is slice, where $-K$ denotes the mirror image of K with reversed orientation.

The knot concordance group \mathcal{C} :

Concordance is an equivalence relation and the concordance classes form an abelian group \mathcal{C} under $\#$ operation.

Cochran, Orr, Teichner (1999) found a filtration of \mathcal{C} :

$$\cdots \subset \mathcal{F}_{(n.5)} \subset \mathcal{F}_{(n)} \subset \cdots \subset \mathcal{F}_{(1.5)} \subset \mathcal{F}_{(1)} \subset \mathcal{F}_{(0.5)} \subset \mathcal{F}_{(0)} \subset \mathcal{C}$$

$\mathcal{F}_{(m)}$ = the set of m -solvable knots which is a subgroup of \mathcal{C} .

Higher order Blanchfield linking forms

COT in 1999 defined

- ▶ a series of groups Γ_n and (noncommutative) rings \mathcal{R}_n
- ▶ $\mathbb{Z}\Gamma_n \subset \mathcal{R}_n \subset \mathcal{K}_n$
- ▶ \mathcal{K}_n denotes the (skew) field of quotients of $\mathbb{Z}\Gamma_n$
- ▶ $\Gamma_0 = \mathbb{Z} = \langle t \rangle$, $\mathcal{R}_0 = \mathbb{Q}[t^{\pm 1}]$, $\mathcal{K}_0 = \mathbb{Q}(t)$.

Consider

- ▶ $\mathcal{A}_0 = H_1(M_K; \mathcal{R}_0)$ the ordinary rational Alexander module
- ▶ $B\ell_0 : \mathcal{A}_0 \times \mathcal{A}_0 \rightarrow \mathcal{K}_0/\mathcal{R}_0$ the Blanchfield form

Higher order Blanchfield linking forms

COT proved

- ▶ An element $a_0 \in \mathcal{A}_0$ and $B\ell_0$ give a homomorphism $\phi_1 : \pi_1(M_K) \rightarrow \Gamma_1$ and an action of $\pi_1(M_K)$ on \mathcal{R}_1
- ▶ One can define $\mathcal{A}_1 = H_1(M_K; \mathcal{R}_1)$ together with a Blanchfield form $B\ell_1 : \mathcal{A}_1 \times \mathcal{A}_1 \rightarrow \mathcal{K}_1/\mathcal{R}_1$
- ▶ Iterating this procedure, given $(a_0, a_1, \dots, a_{n-1})$ with $a_i \in \mathcal{A}_i$, one can define a homomorphism

$$\phi_n : \pi_1(M_K) \rightarrow \Gamma_n$$

and the n -th Alexander module

$$\mathcal{A}_n = H_1(M_K; \mathcal{R}_n)$$

together with the n -th Blanchfield form

$$B\ell_n : \mathcal{A}_n \times \mathcal{A}_n \rightarrow \mathcal{K}_n/\mathcal{R}_n$$

COT obstructions

Suppose K is $(n.5)$ -solvable with $n \geq 1$.

- ▶ For $k \leq n$ and any (a_0, \dots, a_{k-1}) with $a_i \in P_i$, \mathcal{A}_k has a self-annihilating submodule P_k w.r.t. $B\ell_k$.
- ▶ There is $\phi_k = \phi(a_0, \dots, a_{k-1}) : \pi_1(M_K) \rightarrow \Gamma_k$
 \mathcal{A}_k and nonsingular $B\ell_k$ are defined
 $\rho(M_K, \phi_k)$ is defined and equal to zero.

We say that K has *vanishing COT obstructions of order n* if K satisfies these.

It is known that $H_1(M_{K\#J}, \mathcal{R}_k) = H_1(M_K, \mathcal{R}_k) \oplus H_1(M_J, \mathcal{R}_k)$.
However, no guarantee that $P_k^{K\#J} = P_k^K \oplus P_k^J$.

Splittings of ρ -invariants

Theorem. Suppose that $K\#J$ is slice (or $(n.5)$ -solvable).
(So, for $k \leq n$, there is $\phi(a_0, \dots, a_{k-1}) : \pi(M_{K\#J}) \rightarrow \Gamma_k$ and a s.a.s $P_k^{K\#J} \subset H_1(M_{K\#J}; \mathcal{R}_k)$ and $\rho(M_{K\#J}, \phi_k) = 0$.)

Suppose, in addition, for each k , there is a s.a.s $P_k^K \subset H_1(M_K; \mathcal{R}_k)$ such that $P_k^K \subset P_k^{K\#J}$.

Then both K and J have vanishing COT obstructions of order n .

Known Splitting Properties

- ▶ (Levine 1969) Suppose that K_1 and K_2 have coprime Alexander polynomials. If $K_1 \# K_2$ is algebraically slice (Levine invariants vanish), so are both K_1 and K_2 .
- ▶ (K 2001, 2007) Suppose that K_1 and K_2 have coprime Alexander polynomials. If $K_1 \# K_2$ have vanishing Casson-Gordon-Gilmer invariants, so do both K_1 and K_2 .
- ▶ (K & T. Kim 2007) Suppose K_1 and K_2 have coprime Alexander polynomials. If $K_1 \# K_2$ has vanishing COT obstruction of order 1, so do both K_1 and K_2 .

Remark. The additional condition in our theorem is a generalization: (Levine) If K and J have coprime Alexander polynomials, that there is a s.a.s P_0^K such that $P_0^K \subset P_0^{K \# J}$.

Proof of Theorem

1. $P_k^{K\#J} = P_k^K \oplus P_k^J$ since $B\ell_k$ is nonsingular.
2. Splitting property of ρ under connected sum:

$$\rho(M_{K\#J}, \phi) = \rho(M_K, \tilde{\phi}|_{\pi_1(M_K)}) + \rho(M_J, \tilde{\phi}|_{\pi_1(M_J)})$$

3. If a_i was chosen from P_i^K , then $\rho(M_J, \tilde{\phi}|_{\pi_1(M_J)}) = 0$.

Example

Let K be the knot given in COT paper.

K has nonvanishing COT obstruction of order 2.

\mathcal{A}_0 and \mathcal{A}_1 have unique self-annihilating submodules.

Alexander polynomial $(t^{-1} - 3 + t)^2$.

Let J be a knot with Alexander polynomial

$(lt - (l - 1))((l - 1) - lt)$ for $l \in \mathbb{Z}_+$.

Then K and J are not concordant.

Because:

$P_0^{K\#J} = P_0^K \oplus P_0^J$ because Alexander polynomials are coprime.

\mathcal{A}_1 has unique self-annihilating submodule

Using this, one can show $P_1^{K\#J} \cap H_1(M_K, \mathcal{R}_1) = P_1^K$.