Geometric filtrations of link concordance

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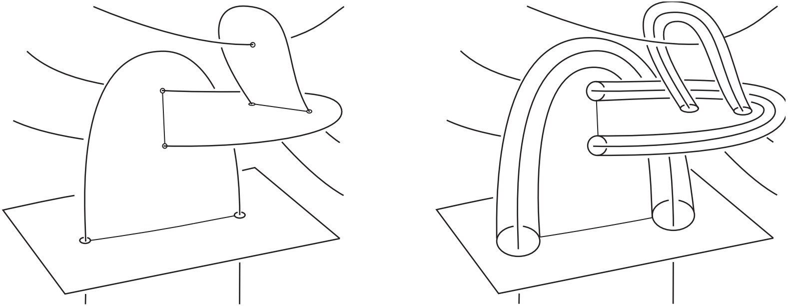


Dec 2009

Goals for this talk

1. Describe a program for determining which links in S^3 bound disjointly embedded gropes of class n in B^4.
2. Show how a tree-valued intersection theory of Whitney towers unifies the Milnor and Arf invariants.
3. Formulate conjectured new invariants of link concordance in terms of framing obstructions on Whitney disks.
4. Advertise a combinatorial conjecture of J. Levine that plays a key role in the program.

Trees

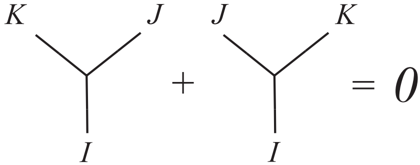
Tree properties:

- All trees are unitrivalent.
- Trivalent vertices are oriented cyclically.
- Univalent vertices are labeled from $\{1, 2, \ldots, m\}$.
- A *rooted tree* has one unlabeled univalent vertex, the *root*.
- All trees graded by *order* $=$ number of trivalent vertices.

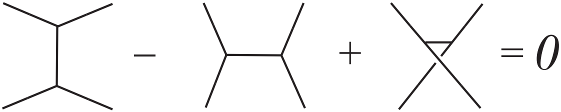
$\mathscr{T}_n$

Definition: $\mathscr{T}_n$ is the abelian group generated by order n trees modulo the following antisymmetry and IHX (local) relations:

AS:

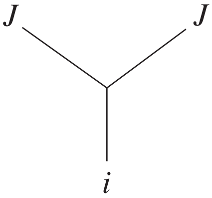


IHX:



Levine Conjecture

The **Levine Conjecture**: $\mathscr{T}_{2n}$ is *torsion-free*, and the torsion subgroup of $\mathscr{T}_{2n-1}$ is generated by the following trees:

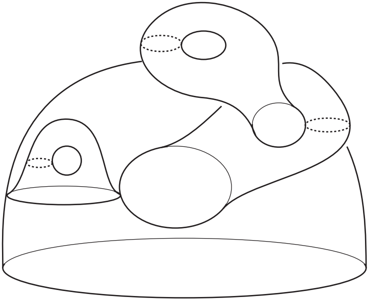


Here $i \in \{1, 2, \ldots, m\}$, and J represents a subtree of order $n-1$.

The Levine Conjecture is <u>true</u> for $n \leq 9$ (and when $m = 2$, for $n \leq 13$).

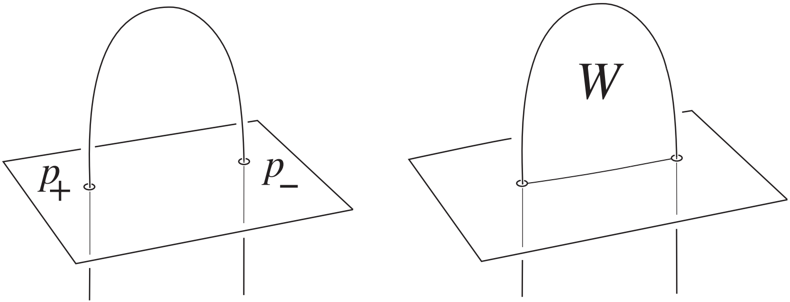
Class 5 grope

Gropes are 'geometric embodiments' of iterated commutators of group elements, with *class* corresponding to commutator length. A class 5 grope:



- It is not known which links in S^3 bound disjointly embedded class n gropes in B^4.
- Will describe a program for computing the class n grope filtration using a related filtration by Whitney towers.

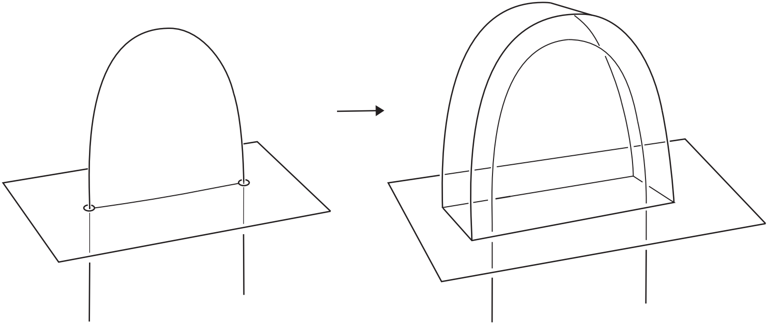
Whitney disks



A canceling pair of intersections p_+ and p_- between two sheets of surfaces in 4-space paired by a *Whitney disk* W.

Whitney move

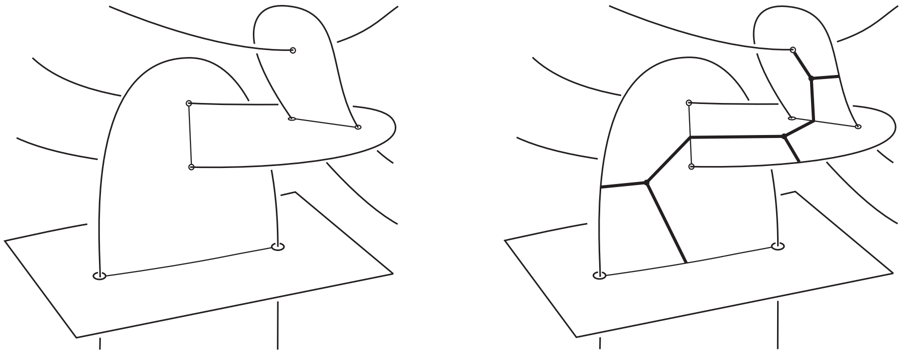
A *Whitney move* can (under favorable circumstances) eliminate a canceling pair of intersections:



But in dimension 4, Whitney disks will not usually be disjointly embedded!

Whitney towers

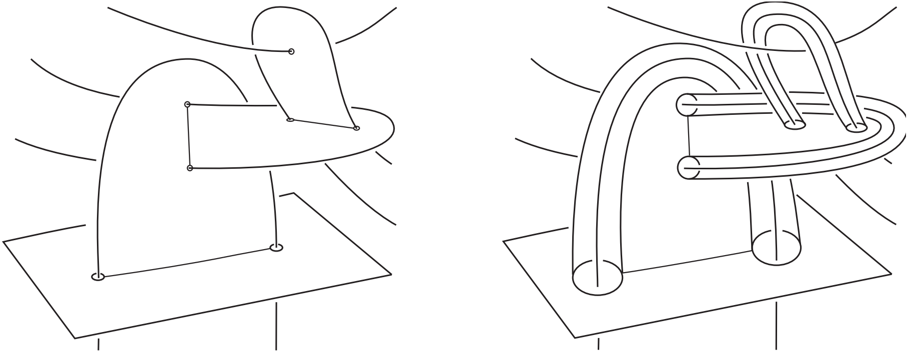
Whitney towers on immersed surfaces in 4-space are built by pairing up *higher order intersections* with *higher order Whitney disks*:



and are parametrized by trees which sit as subsets of the towers.

Whitney towers to gropes

Whitney towers can be converted into gropes:



and <u>order</u> gets converted into <u>class</u>.

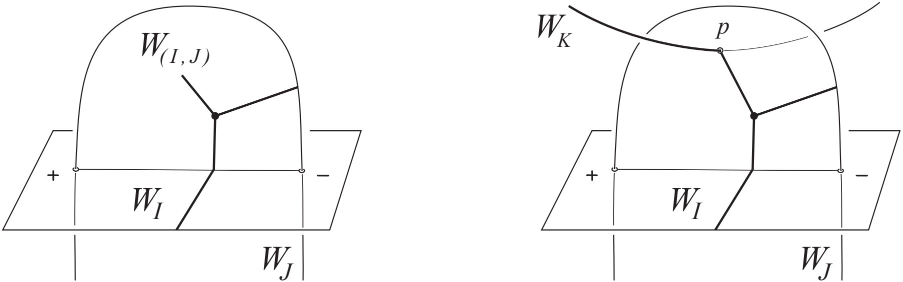
Trees in Whitney towers

- Rooted trees are identified with formal brackets. The *bracket* of two rooted trees I and J is the rooted tree (I, J).
- The *inner product* of two rooted trees I and J is the unrooted tree $\langle I, J \rangle$ gotten by gluing together the roots.

In a Whitney tower:

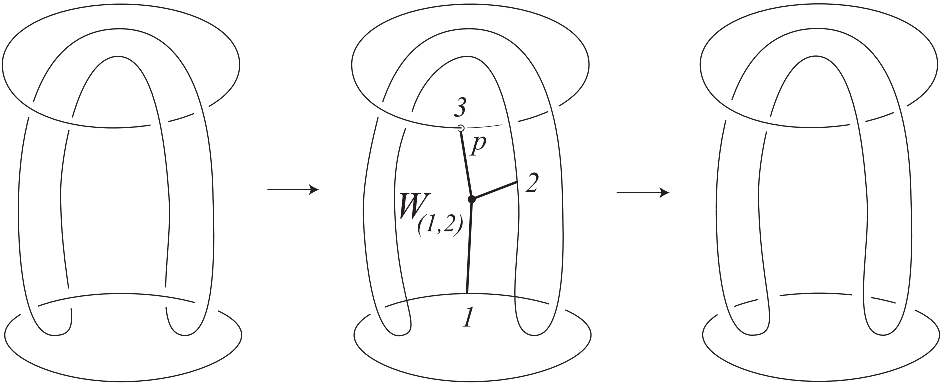
- paired intersections $\longleftrightarrow$ rooted trees
- unpaired intersections $\longleftrightarrow$ unrooted trees

trees and intersections



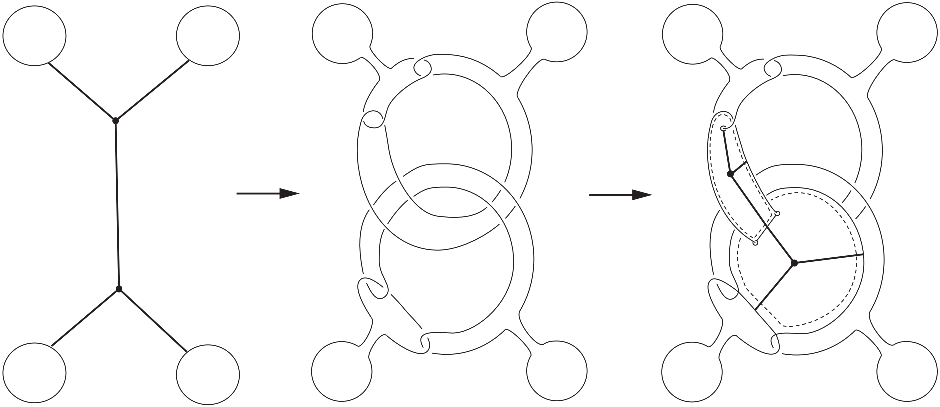
- A Whitney disk $W_{(I,J)}$ contains the rooted tree $(I,J) = \overset{J}{\dashv}_{I}$.
- An intersection point $p \in W_{(I,J)} \cap W_K$ corresponds to the unrooted tree $t_p = \langle (I,J), K \rangle = {}_{K}\overset{J}{\dashv}_{I}$.

Borromean rings



$L = L_1 \cup L_2 \cup L_3$ bounds embedded disks $D_1 \cup D_2 \cup D_3$ in B^4.
Whitney disk $W_{(1,2)}$ pairing $D_1 \cap D_2$ contains rooted tree $(1, 2)$.
$p = W_{(1,2)} \cap D_3$ has associated unrooted tree $t_p = \langle (1, 2), 3 \rangle$.

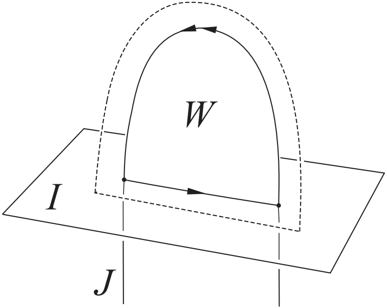
Bing-Cochran-Habiro links realize all intersection trees

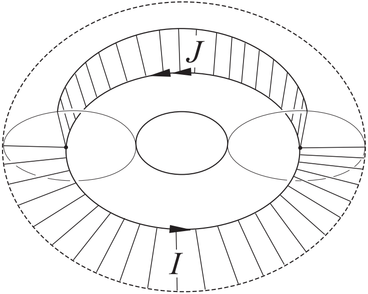


By iterated Bing-doubling, any collection of trees can be realized by a Whitney tower on immersed 2-disks in B^4 bounded by a link $L \subset S^3$.

Twisted Whitney disks

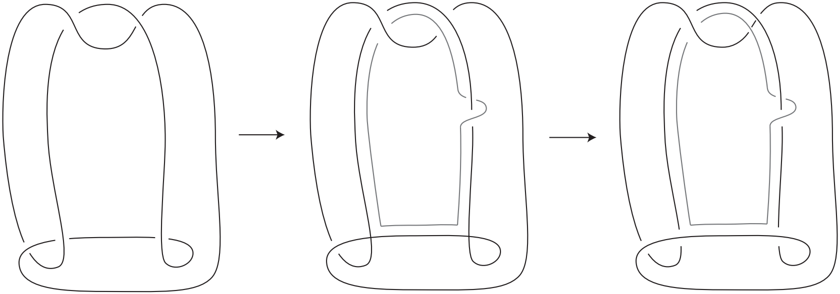
A Whitney disk W has a canonical non-vanishing normal section over ∂W.





- The relative Euler number $\omega(W) \in \mathbb{Z}$ is called the *framing obstruction* of W.
- If $\omega(W) = 0$, W is said to be *framed*, otherwise W is called *twisted*.

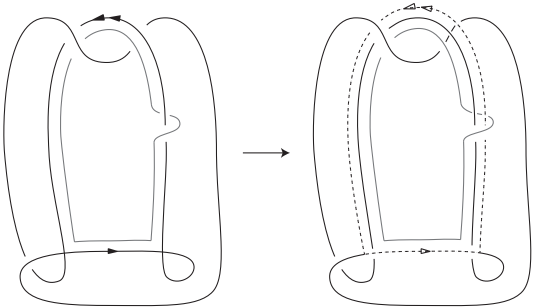
A twisted Whitney disk for the figure-8 knot



Pushing in to the 4-ball from left to right, the figure-8 knot bounds an immersed disk admitting an embedded twisted Whitney disk W.
(The right-most picture is the unlink.)

A twisted Whitney disk for the figure-8 knot

The second and third pictures from the previous frame:



The twisting $\omega(W) = 1$ corresponds to the $+1$-linking between the 'inner' boundary component of a collar on ∂W and a Whitney section.

∞-trees

A twisted Whitney disk W_J corresponding to the rooted tree J is assigned the ∞-tree:

$$q(J) := \infty \text{---} J$$

which is gotten by labeling the root univalent vertex of J with the symbol ∞ (representing a 'twist').

Twisted Whitney towers

An *order* $2n-1$ *twisted Whitney tower* contains only:

- (unrooted) trees of order at least $2n-1$,
- ∞-trees of order at least n.

An *order* $2n$ *twisted Whitney tower* contains only:

- (unrooted) trees of order at least $2n$,
- ∞-trees of order at least n.

Groups of twisted trees

Definition:

$\mathcal{T}^{\infty}_{2n-1}$ is the $\mathbb{Z}$-span of order $2n-1$ trees and order n ∞-trees modulo the relations:

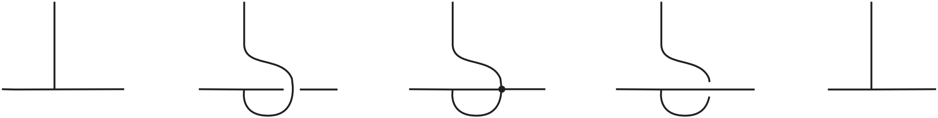
- Antisymmetry and IHX on all trees (including ∞-trees).
- The boundary-twist relation: $q(J,i) = \langle (J,i), J \rangle$.

$\mathcal{T}^{\infty}_{2n}$ is the $\mathbb{Z}$-span of order $2n$ trees and order n ∞-trees modulo the relations:

- Antisymmetry and IHX on all non-∞ trees.
- $q(-J) = q(J)$
- The interior-twist relation: $2q(J) = \langle J, J \rangle$.
- The twisted-IHX relation: $q(I) = q(H) + q(X) - \langle H, X \rangle$.

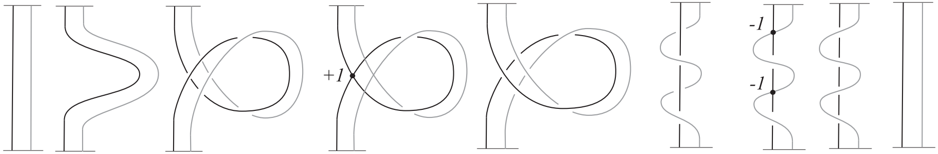
The boundary-twist relation

The boundary-twist relation $q(J,i) = \langle (J,i), J \rangle$ in $\mathscr{T}^{\infty}_{2n-1}$ is realized by a local move in the collar of a Whitney disk:



The interior-twist relation

The interior-twist relation $2q(J) = \langle J, J \rangle$ in $\mathscr{T}^{\infty}_{2n}$ is realized by a local move in the interior of a Whitney disk:



The intersection tree of a twisted Whitney tower

The order n intersection tree $\tau_n^\infty(\mathscr{W})$ of an order n twisted Whitney tower $\mathscr{W}$ is defined by:

$$\tau_n^\infty(\mathscr{W}) \quad = \quad \sum \varepsilon_p \cdot t_p \quad + \quad \sum \omega(W_J) \cdot q(J) \quad \in \mathscr{T}_n^\infty$$

where the first sum is over all order n intersections p (with $\varepsilon_p = \pm 1$), and the second sum is over all Whitney disks W_J.

Raising the order of a Whitney tower

Theorem (S-T 2004, C-S-T 2007, C-S-T 2009)

If a link L bounds an order n twisted Whitney tower $\mathscr{W}$ in B^4 with

$$\tau_n^\infty(\mathscr{W}) = 0 \in \mathscr{T}_n^\infty$$

then L bounds an order $n+1$ twisted Whitney tower.

The proof consists of realizing all relations by controlled geometric constructions.

Does $\tau_n^\infty(\mathscr{W})$ only depend on L?

The first non-vanishing Milnor invariants of a link L

- The *free Lie algebra* $\mathscr{L} = \bigoplus_{n \geq 0} \mathscr{L}_{n+1}$ on generators $\{X_1, X_2, \ldots, X_m\}$ is the free abelian group of rooted trees, modulo self-annihilation  $= 0$ and Jacobi (IHX) relations.
- If the link longitudes $\ell_1, \ldots, \ell_m$ lie in Γ_{n+1}, the $(n+1)$-st term of the lower central series of $\Gamma = \pi_1(S^3 \setminus L)$ then

$$\frac{\Gamma_{n+1}}{\Gamma_{n+2}} \cong \mathscr{L}_{n+1}$$

- The *order n Milnor invariant* $\mu_n(L)$ of L is defined inductively by

$$\mu_n(L) := \sum_{i=1}^{m} [X_i] \otimes [\ell_i] \in \mathscr{L}_1 \otimes \mathscr{L}_{n+1} \cong \mathbb{Z}^m \otimes \mathscr{L}_{n+1}$$

- $\mu_n(L)$ actually lies in $\mathscr{D}_n := \mathrm{Ker}(\mathscr{L}_1 \otimes \mathscr{L}_{n+1} \xrightarrow{Br} \mathscr{L}_{n+2})$.

the η^∞ map

The connection between Whitney towers and Milnor invariants is described by a map $\eta^\infty : \mathcal{T}^\infty_n \to \mathcal{D}_n$ that averages over all ways of making a univalent vertex into a root, e.g.

$$\eta^\infty\left({}_1\overset{3}{\dashv}{}_2 \right) \quad = \quad X_1 \otimes \overset{3}{\dashv}{}_2 \;+\; X_2 \otimes {}_1\overset{3}{\dashv} \;+\; X_3 \otimes {}_1\dashv_2$$

For ∞-trees, η^∞ is defined by:

$$q(J) \mapsto \frac{1}{2}\eta^\infty(\langle J, J \rangle)$$

Theorem (S-T 2003, C-S-T 2009)

If L bounds an order n twisted Whitney tower $\mathcal{W}$, then $\mu_k(L) = 0$ for $k < n$ and

$$\mu_n(L) = \eta^\infty \circ \tau^\infty_n(\mathcal{W})$$

So $\ker(\eta^\infty)$ is last obstruction for Whitney tower and grope filtrations by order and class. (Proof uses tower-to-grope realization of η^∞.)

Obstruction theory

Theorem (C-S-T 2009)

In the setting where the Levine Conjecture holds:

1. *If L bounds an order $(2n-1)$ twisted Whitney tower, then: $\mu_{(2n-1)}(L) = 0$ if and only if L bounds an order $2n$ twisted Whitney tower.*
2. *If L bounds an order $2n$ twisted Whitney tower $\mathscr{W}$, then:*
   - *If n is even, then $\mu_{2n}(L) = 0$ if and only if L bounds an order $2n+1$ twisted Whitney tower.*
   - *If n is odd, then $\mu_{2n}(L) = 0$ if and only if $\tau^{\infty}_{2n}(\mathscr{W}) \in \mathrm{span}\{q(J,J)\} \cong \mathbb{Z}_2 \otimes \mathscr{L}_{\frac{n+1}{2}}$.*

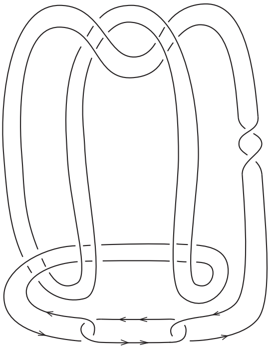
The isomorphism in case 2 of the theorem with n odd sends an ∞-tree $q(J,J)$ to the Lie-bracket determined by the order $\frac{n-1}{2}$ rooted tree J.
Conjecture: $\tau^{\infty}_{2n}(\mathscr{W}) \in \mathbb{Z}_2 \otimes \mathscr{L}_{\frac{n+1}{2}}$ is a non-trivial concordance invariant – the last obstruction to computing the geometric filtrations.

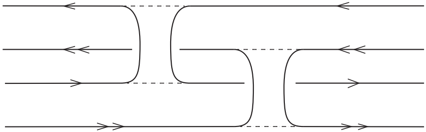
Conjectured new invariants

- Conjecture true for $n=1$: The coefficient of $q(j,j)$ in $\tau_2^\infty(\mathscr{W}) \in \mathrm{span}\{q(j,j)\} \cong \mathbb{Z}_2 \otimes \mathscr{L}_1 \cong \mathbb{Z}_2^m$ is the Arf invariant of the jth component of L.
- **Corollary:** The Arf invariant is the only obstruction for a knot in S^3 to bound a class n grope or order n Whitney tower in B^4, for any n.
- First open case $n=3$ of the conjecture: $L =$ (un-twisted) Bing double of the figure-8 knot.
  L bounds an order 6 twisted Whitney tower $\mathscr{W}$ with non-trivial $\tau_6^\infty(\mathscr{W}) = q((1,2,),(1,2,)) \mapsto [X_1,X_2] \in \mathbb{Z}_2 \otimes \mathscr{L}_2$. (See next slide.)
- All elements of $\mathbb{Z}_2 \otimes \mathscr{L}_{\frac{n+1}{2}}$ for all odd n can be realized by iterated Bing doubles of any knot with non-trivial Arf invariant.
- **Corollary:** Milnor invariants classify the class n grope filtration modulo connect sum with boundary links (assuming the Levine conjecture).

The Bing double of the figure-8 knot

$L =$ Bing(figure-8) is a boundary link:

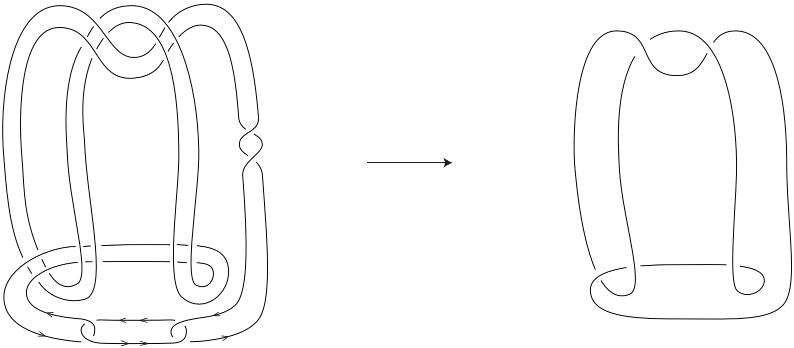




Bing-doubling is realized by the band-sum of pairs of Seifert surfaces as on the right.

So: 'Bing-doubling preserves boundary links.'

The Bing double of the figure-8 knot

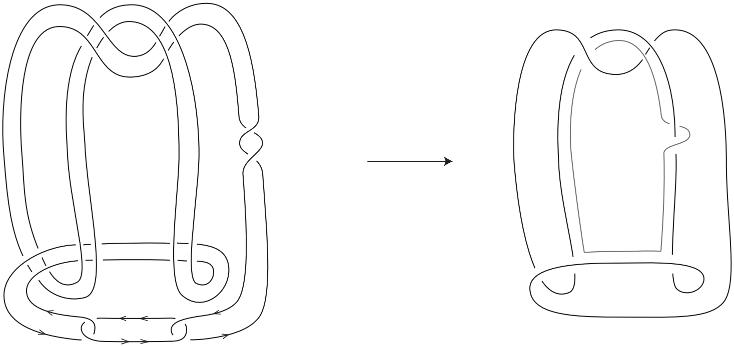


L = Bing(figure-8) bounds an order 6 twisted Whitney tower $\mathscr{W}$:

$$\mathscr{W} = D_1 \cup D_2 \cup W_{(1,2)} \cup W_{((1,2),(1,2))}.$$

$W_{(1,2)}$ is the trace of a null-homotopy of the figure-8 knot.

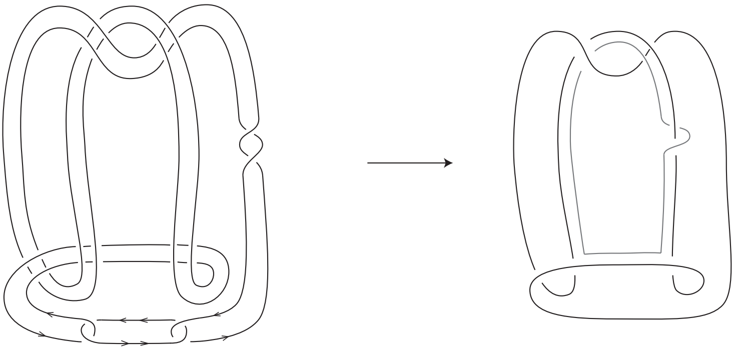
The Bing double of the figure-8 knot



$$\tau_6^{\infty}(\mathscr{W}) = q((1,2),(1,2)) \mapsto [X_1, X_2] \neq 0 \in \mathbb{Z}_2 \otimes \mathscr{L}_2$$

(Since $W_{((1,2),(1,2))}$ is clean, and $\omega(W_{((1,2),(1,2))}) = 1$).

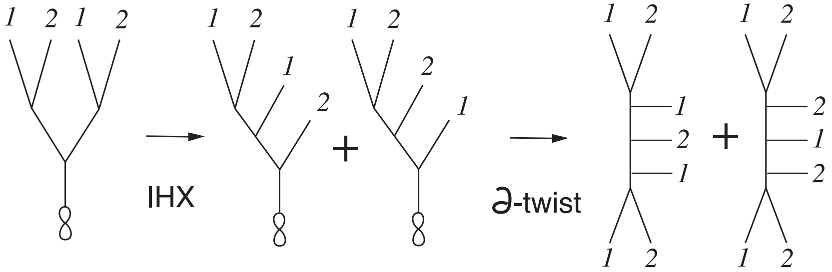
The Bing double of the figure-8 knot



The conjecture states that L does not bound an order 7 twisted Whitney tower or class 7 disjointly embedded gropes in B^4.
L is known not to be slice (J. C. Cha).

Why twisted Whitney disks?

Using IHX and boundary-twist moves, can change the Whitney tower bounded by L, eliminating all twisted Whitney disks:



Each of the two trees on the right map *non-trivially* to Milnor invariants! Their *sum* lies in $\ker \eta^\infty$, as captured geometrically by the original left-most symmetric twisted Whitney disk.