

Homology cylinders in knot theory

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and

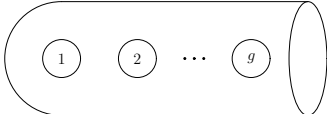
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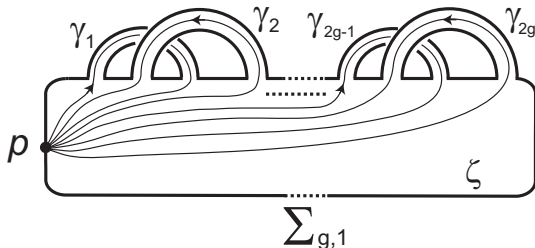
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Introduction (1)

§1. Introduction

- $\Sigma_{g,1} =$  $(g \geq 0, \text{ oriented})$

with a standard cell decomposition:



Definition (Goussarov, Habiro, Garoufalidis-Levine, Levine)

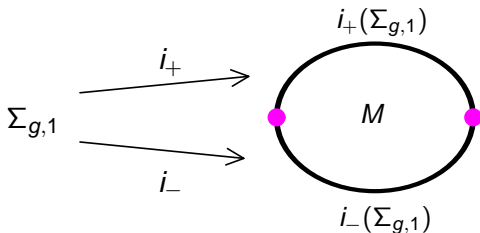
(M, i_+, i_-) : a *homology cylinder* (HC) over $\Sigma_{g,1}$

$\stackrel{\text{def}}{\iff} \begin{cases} M : \text{a compact oriented 3-manifold,} \\ i_+, i_- : \Sigma_{g,1} \hookrightarrow \partial M \text{ two embeddings (markings)} \end{cases}$

satisfying

- 1 i_+ : orientation-preserving, i_- : orientation-reversing;
- 2 $\partial M = i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,1})$,
 $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1}) = i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$;
- 3 $i_+|_{\partial\Sigma_{g,1}} = i_-|_{\partial\Sigma_{g,1}}$;
- 4 $i_+, i_- : H_*(\Sigma_{g,1}; \mathbb{Z}) \xrightarrow{\cong} H_*(M; \mathbb{Z})$ isomorphisms.

- (M, i_+, i_-) : a homology cylinder (over $\Sigma_{g,1}$)



Definition

$$\mathcal{C}_{g,1} := \{(M, i_+, i_-) : \text{HC over } \Sigma_{g,1}\} / (\text{marking pres. diffeo}).$$

Stacking

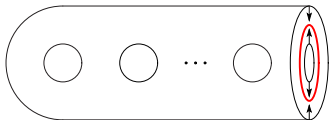
For $(M, i_+, i_-), (N, j_+, j_-) \in \mathcal{C}_{g,1}$,

$$(M, i_+, i_-) \cdot (N, j_+, j_-) := (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-) \in \mathcal{C}_{g,1}$$

$\leadsto \mathcal{C}_{g,1}$ becomes a **monoid**.

unit: $(\Sigma_{g,1} \times [0, 1], \text{id} \times 1, \text{id} \times 0)$

where corners of $\Sigma_{g,1} \times [0, 1]$ are rounded, and



Examples

① $\mathcal{M}_{g,1}$: the mapping class group of $\Sigma_{g,1}$

$[\varphi] \in \mathcal{M}_{g,1}$, i.e.

$\varphi : \Sigma_{g,1} \xrightarrow{\sim} \Sigma_{g,1}$: a diffeo. s.t. $\varphi|_{\partial\Sigma_{g,1}} = \text{id}$

$\implies (\Sigma_{g,1} \times I, \text{id} \times 1, \varphi \times 0) \in \mathcal{C}_{g,1}$.

We can check

$\mathcal{M}_{g,1} \hookrightarrow \mathcal{C}_{g,1}$: monoid embedding

$\rightsquigarrow \mathcal{C}_{g,1}$ is an *enlargement* of $\mathcal{M}_{g,1}$.

- ② surgery along **clovers** (Goussarov) or **claspers** (Habiro)
- ③ surgery along **pure string links** (Habegger, Levine)
- ④ connected sum with a homology 3-sphere X :
 $(\Sigma_{g,1} \times [0, 1] \# X, \text{id} \times 1, \text{id} \times 0) \in \mathcal{C}_{g,1}$.

Today we focus on

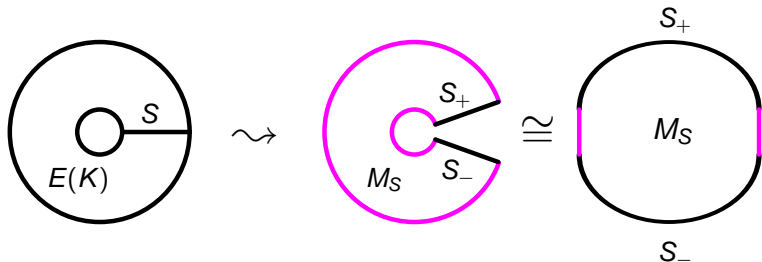
- ⑤ complementary sutured manifolds of Seifert surfaces of **a special class of knots**.

Homological fibered knots (1)

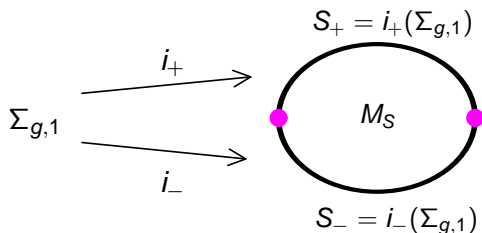
§2. Homological fibered knots

- $K \subset S^3$: a knot,
 S : a Seifert surface of K of genus g .

M_S : the cobordism obtained from $E(K)$ by cutting along S
= the **(complementary) sutured manifold** for S .



- By using any identification $i : \Sigma_{g,1} \xrightarrow{\cong} S$, we obtain a *marked* sutured manifold (M_S, i_+, i_-) :



Question When this becomes a homology cylinder?

Proposition (Crowell-Trotter, ..., Goda-S.)

K : a knot in S^3 ,

K has a Seifert surface S of genus g s.t. M_S is a homology product (over a copy of S)

\iff The following hold:

- S is a minimal genus Seifert surface,
- The Alexander polynomial $\Delta_K(t)$ of K is monic,
- $\deg(\Delta_K(t)) = 2 \text{ genus}(K)$.

Definition

A knot K in S^3 is said to be *homologically fibered* if

- (1) $\Delta_K(t)$ is monic,
- (2) $\deg(\Delta_K(t)) = 2 \text{ genus}(K)$.

Remarks

- (Fibered knots) \subset (HFknots [Homological Fibered knots]) corresponds to $\mathcal{M}_{g,1} \subset \mathcal{C}_{g,1}$.
- We can define *rational* homological fibered knots (\mathbb{Q} -HFknot) by assuming only (2).

“Uniqueness”

Proposition

K : an HFknot of genus g

S_1, S_2 : minimal genus Seifert surfaces

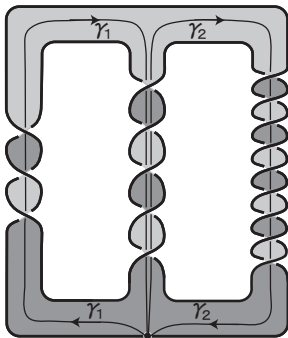
For any markings of ∂M_{S_1} and ∂M_{S_2} , $\exists N \in \mathcal{C}_{g,1}$ s.t.

$$M_{S_1} \cdot N = N \cdot M_{S_2} \in \mathcal{C}_{g,1}$$

In particular, any monoid homomorphism

$$\mathcal{C}_{g,1} \rightarrow A \quad \text{w/ } A: \text{ an abelian group}$$

gives an invariant of HFknots.



Pretzel knot $P(-3, 5, 9)$ is an HFknot.

Easy to see $P(-2n + 1, 2n + 1, 2n^2 + 1)$ is an HFknot.

§3. Factorization formulas of Alexander invariants

Classical case

- $K \subset S^3$: a knot,
 S : a Seifert surface of K w/ a Seifert matrix A .

Assume that A is invertible over \mathbb{Q} (i.e. K is a \mathbb{Q} -HFknot).

Then

$$\begin{aligned}\Delta_K(t) &= \det(A^T - tA) \\ &= \det(A^T) \det(I_{2g(S)} - t(A^T)^{-1}A)\end{aligned}$$

What does this *factorization* mean?

We can check:

- A^T and A represent

$$i_+, i_- : \mathbb{Z}^{2g} \cong H_1(\Sigma_{g,1}) \longrightarrow H_1(M_S) \cong \mathbb{Z}^{2g}$$

under certain bases of $H_1(\Sigma_{g,1})$ and $H_1(M_S)$. In fact,

$$\begin{aligned} \det(A) &= \text{The top (bottom) coeff. of } \Delta_K(t) \\ &= \pm |H_1(M, i_+(\Sigma_{g,1}))| \\ &= \tau(C_*(M_S, i_+(\Sigma_{g,1}); \mathbb{Q}) \quad \text{torsion} \end{aligned}$$

- $\sigma(M_S) := (A^T)^{-1}A \in Sp(2g, \mathbb{Q})$.
(Can regard $\sigma(M_S)$ as an H_1 -monodromy of M_S .)

So, roughly speaking, our factorization formula says

$$\begin{aligned}\Delta_K(t) &= \det(A^T) \det(I_{2g(S)} - t(A^T)^{-1}A) \\ &= (\text{torsion of } M_S) \cdot (\text{effect of } H_1\text{-monodromy of } M_S).\end{aligned}$$

Remark By Milnor,

$$\frac{\Delta_K(t)}{1-t} = \tau_{\mathbb{Z}}(K),$$

where $\tau_{\mathbb{Z}}(K)$ is the **Reidemeister torsion** associated with the \mathbb{Z} -cover of $E(K)$.

- For an HFknot K ,

$$\begin{aligned}\Delta_K(t) &= \det(A^T) \det(I_{2g(S)} - t(A^T)^{-1}A) \\ &= \pm \det(I_{2g(S)} - t(A^T)^{-1}A).\end{aligned}$$

~> The factorization formula is useless for HFknots!

~> We will give a generalization by using twisted homology.

Higher-order case (Twisted coefficients)

- K : an HFknot,
- $M_S = (M_S, i_+, i_-) \in \mathcal{C}_{g,1}$: an HC associated with K ,
- $\mathcal{K} := \text{Frac}(\mathbb{Z}H_1(M_S)) \cong \mathbb{Q}(t_1, \dots, t_{2g})$ as twisted coefficients.

Lemma

For $\pm \in \{+, -\}$, $H_*(M_S, i_\pm(\Sigma_{g,1}); \mathcal{K}) = 0$.

cf. classical case: $H_*(M_S, i_\pm(\Sigma_{g,1}); \mathbb{Z}) = 0$.

Definition

- The **\mathcal{K} -torsion** $\tau_{\mathcal{K}}(M_S)$ is

$$\tau_{\mathcal{K}}(M_S) := \tau(C_*(M_S, i_+(\Sigma_{g,1}); \mathcal{K})) \in GL(\mathcal{K}) / \sim .$$

- The **Magnus matrix** $r_{\mathcal{K}}(M_S) \in GL(2g, \mathcal{K})$ is the representation matrix of the right \mathcal{K} -isom.:

$$\begin{array}{ccc}
 H_1(\Sigma_{g,1}, \mathcal{K}) & \xrightarrow[i_-]{\cong} & H_1(M_S, \mathcal{K}) & \xrightarrow[i_+^{-1}]{\cong} & H_1(\Sigma_{g,1}, \mathcal{K}) \\
 \parallel & & & & \parallel \\
 \mathcal{K}^{2g} & \xrightarrow[r_{\mathcal{K}}(M_S)]{\cong} & & & \mathcal{K}^{2g}
 \end{array}$$

Remark By substituting $t_i \mapsto 1$, we have

$$\tau_{\mathcal{K}}(M_S) \mapsto \det A = \pm 1, \quad r_{\mathcal{K}}(M_S) \mapsto \sigma(M_S).$$

- If K is fibered w/ the monodromy $\varphi \in \mathcal{M}_{g,1}$, then

$$r_{\mathcal{K}}(M_S) = \overline{\left(\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i} \right)_{1 \leq i, j \leq 2g}}.$$

Theorem (Fibered obstructions)

K, M_S : as before.

If K is **fibered**, then

- 1 all the entries of the Magnus matrix $r_{\mathcal{K}}(M_S)$ are **Laurent polynomials** in $\mathbb{Q}[t_1^{\pm}, \dots, t_{2g}^{\pm}] \subset \mathcal{K} = \mathbb{Q}(t_1, \dots, t_{2g})$,
- 2 the \mathcal{K} -torsion $\tau_{\mathcal{K}}(M_S)$ is **trivial**.

Theorem (Factorization formula)

Let

$$\rho : \pi_1(E(K)) \longrightarrow \frac{\pi_1(E(K))}{\pi_1(E(K))''} \cong H_1(M_S) \rtimes H_1(E(K))$$

be the natural projection, and let $t \in H_1(E(K))$ be a meridian loop. Then we have

$$\begin{aligned} \tau_{\mathcal{K}(t^\pm; \sigma)}(E(K)) &= \frac{\tau_{\mathcal{K}}(M_S) \cdot (I_{2g} - t \cdot r_{\mathcal{K}}(M_S))}{1 - t} \\ &\in \mathcal{K}_1(\mathcal{K}(t^\pm; \sigma)) / \pm \rho(\pi_1(E(K))), \end{aligned}$$

where LHS is the **noncommutative higher-order torsion** associated with ρ (defined by Cochran, Harvey and Friedl).

§4. 12-crossings non-fibered homological fibered knots

Facts on fibered knots vs. HFknots

- HFknots with at most 11-crossings are all fibered.
- There are 13 non-fibered HFknots with 12-crossings. In particular, Friedl-Kim showed that these 13 knots are not fibered by using twisted Alexander polynomial associated with finite representations.

We computed $r_{\mathcal{K}}(M_S)$ and $\tau_{\mathcal{K}}(M_S)$ for these 13 knots and checked that **each of them also detects the non-fiberedness of all 13 HFknots.**

Recipe

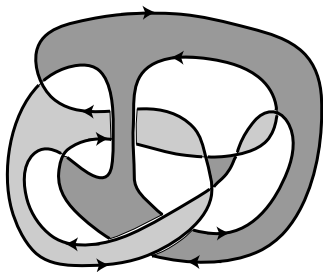
- 1 Get all the pictures of those 13 knots.
[By Computer (Database (KnotInfo) on Internet)]
- 2 For each of them,
 - 1 Find a minimal genus Seifert surface S .
[By hand]
 - 2 Calculate an **admissible** presentation of $\pi_1(M_S)$.
[By hand]
 - 3 Compute $r_{\mathcal{K}}(M_S)$ and $\tau_{\mathcal{K}}(M_S)$.
[By hand and also by computer program]

12-crossings non-fibered homological fibered knots (3)

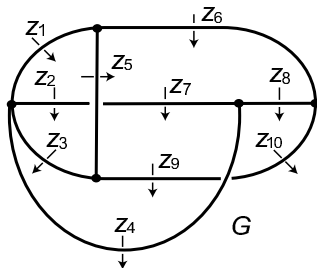
① List of non-fibered HFknots:

Knot 12n-	Genus	Alexander polynomial
0057	2	$1 - 2t + 3t^2 - 2t^3 + t^4$
0210	3	$1 - t - t^2 + 3t^3 - t^4 - t^5 + t^6$
0214	3	$1 - t - t^2 + 3t^3 - t^4 - t^5 + t^6$
0258	2	$1 - 4t + 5t^2 - 4t^3 + t^4$
0279	2	$1 - 6t + 11t^2 - 6t^3 + t^4$
0382	2	$1 - 5t + 7t^2 - 5t^3 + t^4$
0394	2	$1 - 6t + 11t^2 - 6t^3 + t^4$
0464	2	$1 - 4t + 5t^2 - 4t^3 + t^4$
0483	2	$1 - 4t + 5t^2 - 4t^3 + t^4$
0535	2	$1 - 7t + 11t^2 - 7t^3 + t^4$
0650	2	$1 - 4t + 7t^2 - 4t^3 + t^4$
0801	2	$1 - 5t + 7t^2 - 5t^3 + t^4$
0815	2	$1 - 2t + t^2 - 2t^3 + t^4$

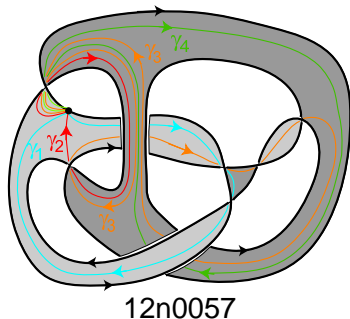
- 2 Example of calculation of admissible presentation



12n0057

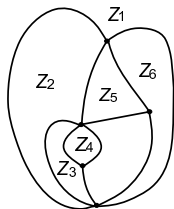
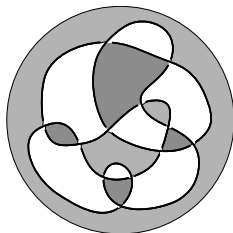


12-crossings non-fibered homological fibered knots (5)

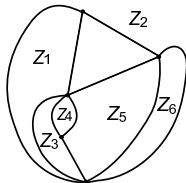
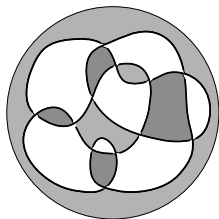


Generators	$i_-(\gamma_1), \dots, i_-(\gamma_4), z_1, \dots, z_{10}, i_+(\gamma_1), \dots, i_+(\gamma_4)$
Relations	$z_1 z_5 z_6^{-1}, z_2 z_3 z_4 z_1, z_3 z_9^{-1} z_5^{-1}, z_7 z_4 z_8^{-1}, z_8 z_{10} z_6,$ $z_2 z_5 z_7^{-1} z_5^{-1}, z_9 z_4 z_{10}^{-1} z_4^{-1}, i_-(\gamma_1) z_1^{-1} z_5^{-1}, i_-(\gamma_2) z_2,$ $i_-(\gamma_3) z_4 z_8 z_7 z_5^{-1}, i_-(\gamma_4) z_4, i_+(\gamma_1) z_5^{-1}, i_+(\gamma_2) z_9^{-1} z_6^{-1},$ $i_+(\gamma_3) z_6 z_4 z_7 z_5^{-1} z_3^{-1} z_5 z_6^{-1}, i_+(\gamma_4) z_6 z_7^{-1} z_6^{-1}$

12-crossings non-fibered homological fibered knots (6)

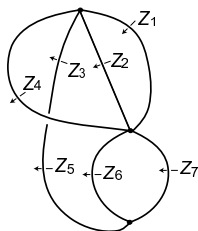
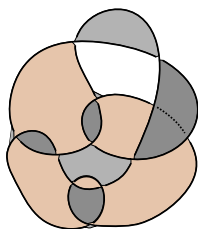


12n0210

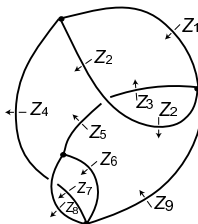
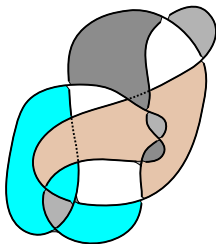


12n0214

12-crossings non-fibered homological fibered knots (7)

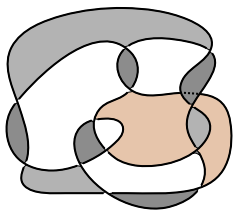


12n0258

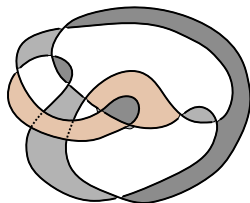
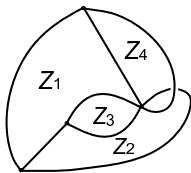


12n0279

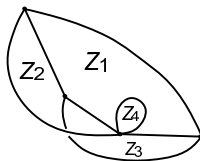
12-crossings non-fibered homological fibered knots (8)



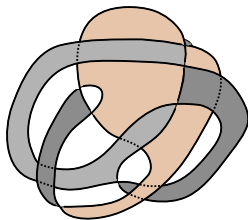
12n0382



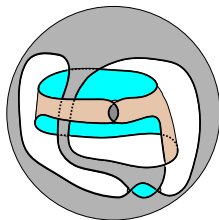
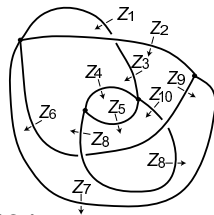
12n0394



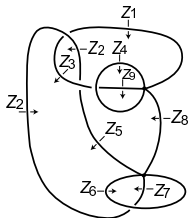
12-crossings non-fibered homological fibered knots (9)



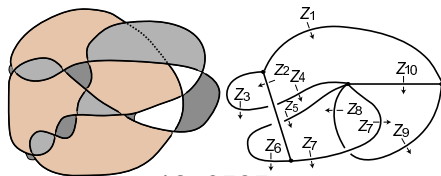
12n0464



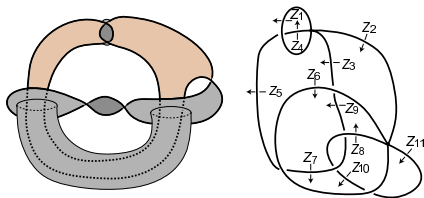
12n0483



12-crossings non-fibered homological fibered knots (10)

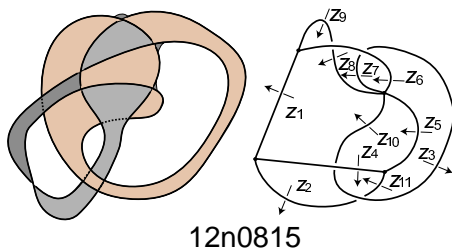
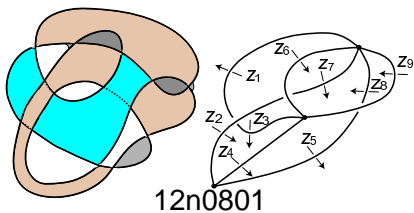


12n0535



12n0650

12-crossings non-fibered homological fibered knots (11)



Computational results for 12n0057

$$r_{\mathcal{K}}(M_S) = \begin{pmatrix} \frac{x^3+x^1 x^2^2 (-1+x^2 (-1+x^4)) - x^2 x^3 x^4}{x^1 x^2^2 (-1+x^2 (-1+x^4))} & - \frac{(-1+x^4) (-1+x^2 x^4)}{-1+x^2 (-1+x^4)} & \frac{x^4}{1+x^2-x^2 x^4} & 0 \\ - \frac{(1+x^1 x^2) x^3}{x^1^2 x^2 (-1+x^2 (-1+x^4))} & - \frac{x^2 (1+x^1 x^2) (-1+x^4)}{x^1 (-1+x^2 (-1+x^4))} & - \frac{(1+x^2) (1+x^1 x^2^2 (-1+x^4))}{x^1 x^2 (-1+x^2 (-1+x^4))} & \frac{1}{x^4} \\ \frac{x^3}{x^1 (-1+x^2 (-1+x^4))} & \frac{x^2^2 (-1+x^4)}{-1+x^2 (-1+x^4)} & \frac{x^2 (1+x^2) (-1+x^4)}{-1+x^2 (-1+x^4)} & 0 \\ \frac{(x^1 x^2^2 - x^3) x^4}{x^1^2 x^2 (-1+x^2 (-1+x^4))} & \frac{x^2 x^4 (x^1 x^2 + x^3 - x^3 x^4)}{x^1 x^3 (-1+x^2 (-1+x^4))} & \frac{(1+x^2) (x^1 x^2^2 - x^3) x^4}{x^1 x^2 x^3 (-1+x^2 (-1+x^4))} & 1 \end{pmatrix},$$

$$\tau_{\mathcal{K}}(M_S) = x^1 x^2^4 + x^1 x^2^5 - x^1 x^2^5 x^4,$$

where $x^j = i_+(\gamma_j)$.

Each of $r_{\mathcal{K}}(M_S)$ and $\tau_{\mathcal{K}}(M_S)$ shows that 12n0057 is not fibered!

§5. Further results and projects

- Detection of non-fiberedness by Johnson-Morita homomorphism:
(2nd Johnson homomorphism + *Yokomizo cokernel*)
- Local moves among HFknots???
- *Categorification* of factorization formulas???

$$\begin{array}{ccc} \widehat{HFK}(K) & \rightsquigarrow & \Delta_K(t), \\ & \begin{array}{c} ? \\ \rightsquigarrow \end{array} & \tau_{\mathcal{K}(t^{\pm}; \sigma)}(E(K)), \\ SFH(M_S, K) & \rightsquigarrow & \tau_{\mathcal{K}}(M_S), \\ ??? & \rightsquigarrow & r_{\mathcal{K}}(M_S). \end{array}$$

decategorification

Fin