Parity criterion for unstabilized Heegaard splittings

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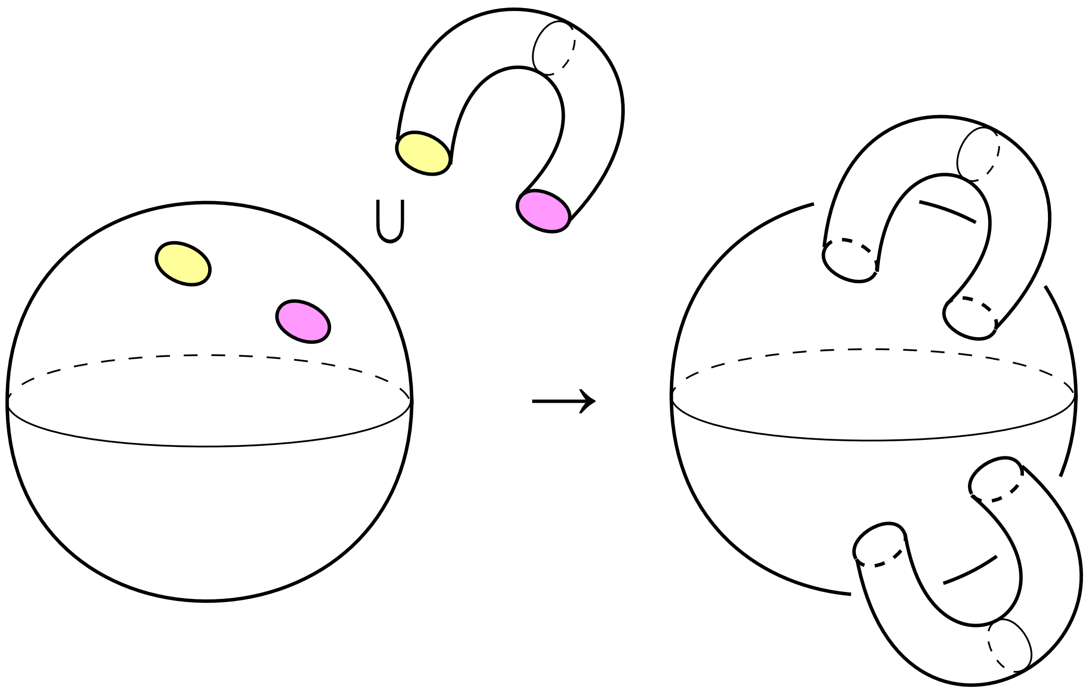
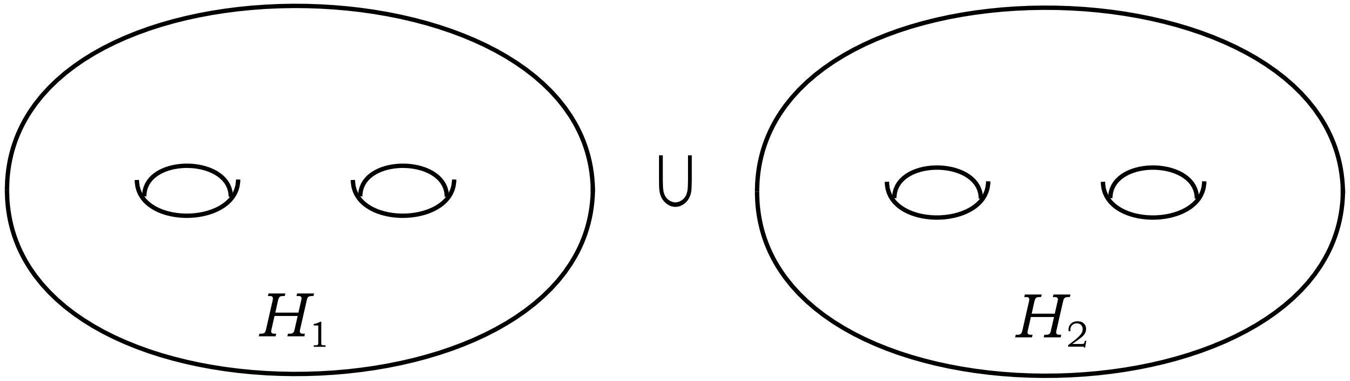
- **handlebody**



A handlebody can be obtained from a 3-ball by attaching 1-handles.

• Heegaard splitting

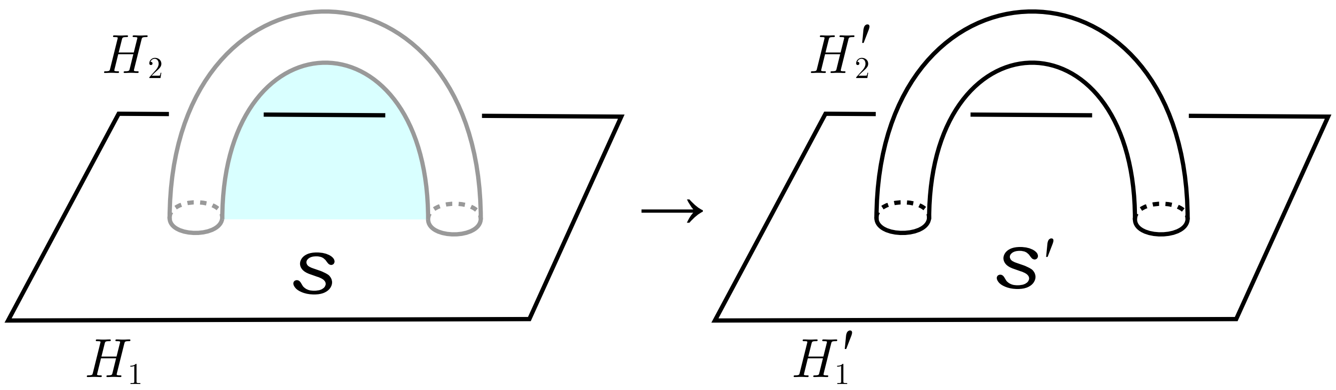
A **Heegaard splitting** $M = H_1 \cup_S H_2$ of a closed 3-manifold M is a decomposition of M into two handlebodies H_1 and H_2. $(S = \partial H_1 = \partial H_2)$



Every compact 3-manifold admits Heegaard splittings.

• stabilization

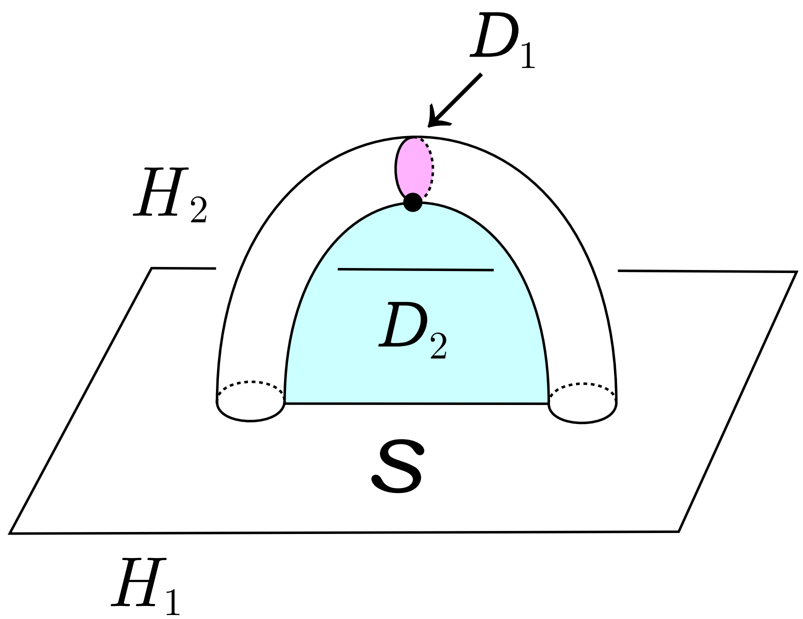
Add a trivial 1-handle to H_1 and remove it from H_2.



This results in a new Heegaard splitting $H_1' \cup_{S'} H_2'$ with genus increased by one.

Equivalently,

A Heegaard splitting $H_1 \cup_S H_2$ obtained by a stabilization has essential disks $D_1 \subset H_1$ and $D_2 \subset H_2$ with $|D_1 \cap D_2| = 1$.



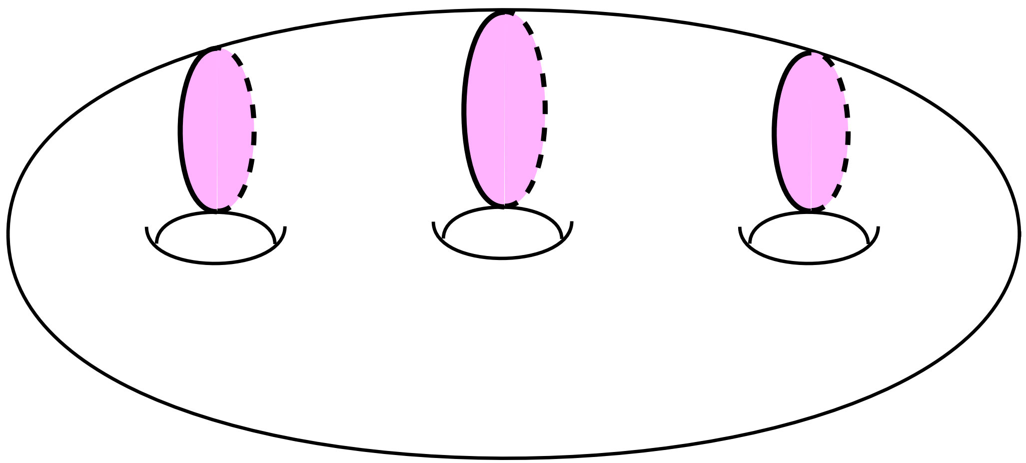
Example

[Waldhausen] Any positive genus Heegaard splitting of S^3 is stabilized.

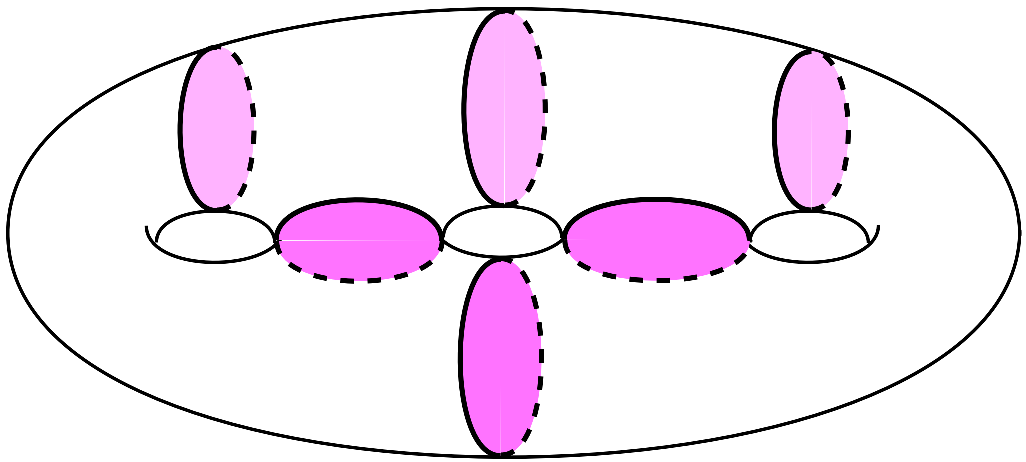
We are interested in Heegaard splittings which are not stabilized. (unstabilized Heegaard splittings.)

For a genus $g \geq 2$ handlebody H,

a collection of essential disks $\{D_1, \cdots, D_g\}$ in H is called a **complete meridian disk system** if the result of cutting H along $\bigcup_{i=1}^{g} D_i$ is a 3-ball.



We say that
a collection of mutually disjoint essential disks $\{D_1, \cdots, D_{3g-3}\}$ in H gives a **pants decomposition** if $\bigcup_{i=1}^{3g-3} \partial D_i$ cuts ∂H into a collection of $2g-2$ pair of pants.



Theorem A

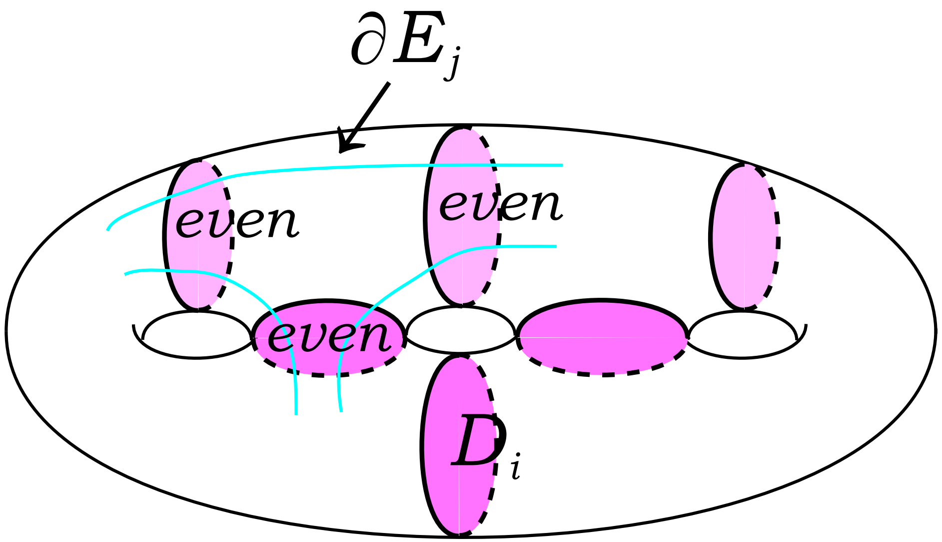
Let $M = H_1 \cup_S H_2$ be a genus $g \geq 2$ Heegaard splitting of a 3-manifold M and $\{D_1, \cdots, D_g\}$ and $\{E_1, \cdots, E_g\}$ be **complete meridian disk systems** of H_1 and H_2, respectively.

If $|D_i \cap E_j| \equiv 0 \pmod 2$ for all the pairs (i, j), then $H_1 \cup_S H_2$ is **unstabilized**.

Lemma

Suppose that $\{D_1, \cdots, D_g\}$ and $\{E_1, \cdots, E_g\}$ satisfy that $|D_i \cap E_j| \equiv 0 \pmod 2$ for all $1 \leq i, j \leq g$.

Then there exist **pants decomposition** $\{D_1, \cdots, D_g, D_{g+1}, \cdots, D_{3g-3}\}$ of H_1 and $\{E_1, \cdots, E_g, E_{g+1}, \cdots, E_{3g-3}\}$ of H_2 such that $|D_i \cap E_j| \equiv 0 \pmod 2$ for all $1 \leq i, j \leq 3g-3$.



Theorem A'

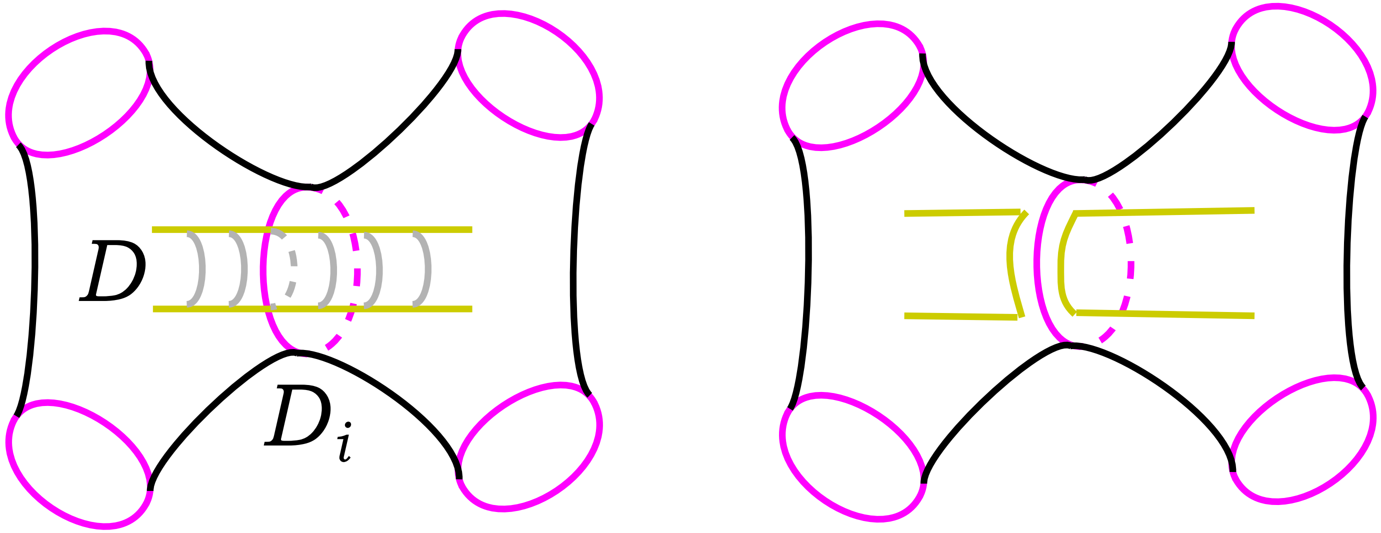
Let $M = H_1 \cup_S H_2$ be a genus $g \geq 2$ Heegaard splitting of a 3-manifold M and $\{D_1, \cdots, D_{3g-3}\}$ and $\{E_1, \cdots, E_{3g-3}\}$ give **pants decomposition** of H_1 and H_2, respectively.

If $|D_i \cap E_j| \equiv 0 \pmod 2$ for all $1 \leq i, j \leq 3g-3$, then $H_1 \cup_S H_2$ is **unstabilized**.

Sketch of proof)

Suppose it is stabilized. Then there exists disks D in H_1 and E in H_2 such that $|D \cap E| = 1$.

Cut D by $\bigcup_{i=1}^{3g-3} D_i$ into subdisks and connect endpoints of arcs in S as in the Figure. Do the same for E with $\bigcup_{j=1}^{3g-3} E_j$. Then the parity of number of intersections of new curves is not changed.

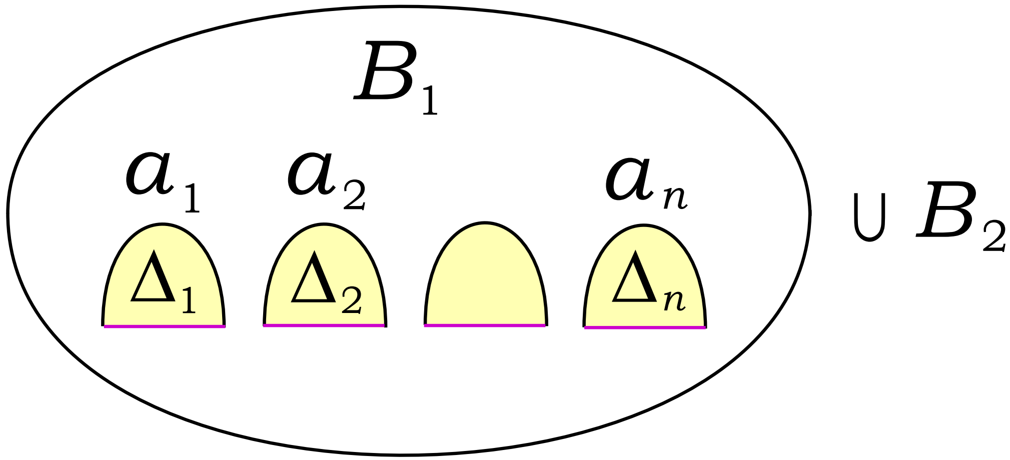


Now $|\partial D \cap \partial E|$ is equivalent to $\sum |\partial D_i \cap \partial E_j|$ in (mod2) and by the hypothesis of Theorem A', it is 0 (mod2).

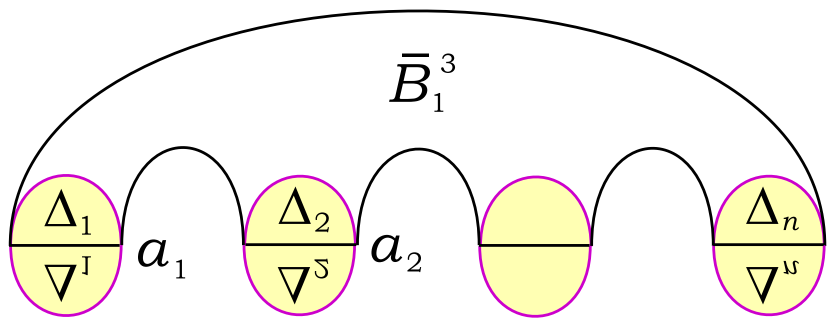
This contradicts that $|D \cap E| = 1$.

• Double (2-fold) branched covering

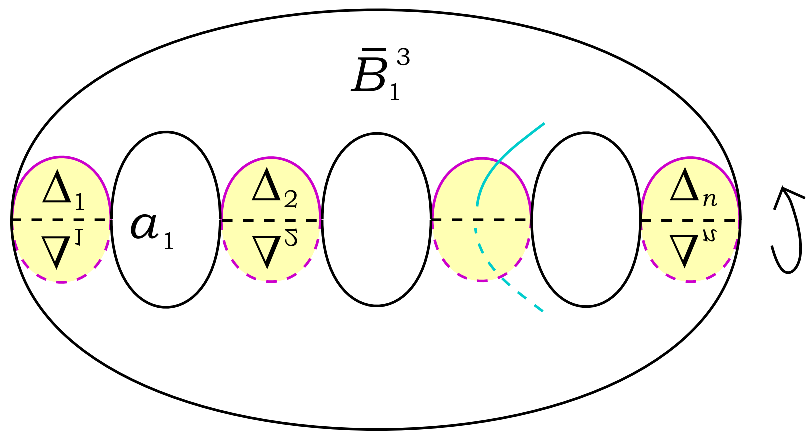
Let L be an n-bridge link in S^3.
Let $\{a_i\}$ be a collection of bridge arcs and Δ_i's be the corresponding disks in a 3-ball B_1 in an n-bridge presentation of L.



Cut B_1 along $\bigcup \Delta_i$ to get $\bar{B}_1$.



Double $\bar{B}_1^3$ along Δ_i's.



$\Longrightarrow$ We obtain a genus $n-1$ Heegaard splitting of the 2-fold branched cover of S^3 over L

• **application**

• **Theorem**

Let L be an n-component, n-bridge link in S^3.
Then the induced Heegaard splitting of the 2-fold branched cover of S^3 over L is unstabilized.

Remark There are examples (**e.g.** (among) torus knots) such that the Heegaard splitting for the 2-fold cover induced from minimal bridge presentation is stabilized.