A Characterization of Heptagonal Knots

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Knots and links in graphs

- Consider a graph as a topological object
- [Conay-Gordon]
  Every embedding of K_6 into $\mathbb{R}^3$ contains a non-splittable 2-component link as a pair of its cycles.
  Furthermore, every embedding of K_7 into $\mathbb{R}^3$ contains a non-trivial knot as a cycle.
- An embedding of a graph into $\mathbb{R}^3$ is linear if each edge is sent to a line segment.
- [Negami]
  Given a link L, there exists a number $r(L)$ s.t. every linear embedding of K_n with $n \geq r(L)$ contains a link of type L.
- For example, $r(\text{trefoil}) = 7$ [Alfonsín]

Knots and Links in linear embeddings of K_6

- Polygon index, $p(K)$: minimal number of line segments necessary for realizing the knot type of K as a polygon.

- $p(\text{unknot}) = 3$, $p(3_1) = 6$, $p(4_1) = 7$,
  and $p(K) \geq 8$ for any other knot type K.

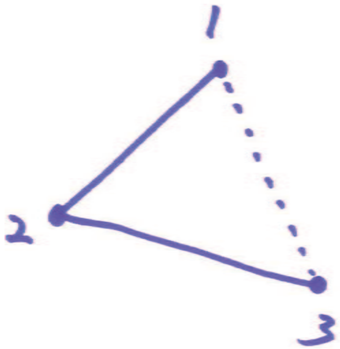
- Theorem 1. [Huh-Jeon]
  (i) Any linear embedding of K_6 contains at most one trefoil knot
  (ii) Only one Hopf link $\Leftrightarrow$ No trefoil knot
  (iii) Three Hopf links $\Leftrightarrow$ Only one trefoil knot

Proof of Theorem 1-(i):

Let P be a hexagonal trefoil knot with vertices $\{1, 2, \ldots, 6\}$. Can be assumed the vertices are in general position.

Then any triangle of P is penetrated by some line segment of P. For example the triangle 123 is pierced by 45 or 56.

WLOG assume 56 does. Then we can prove that the following table holds for P.



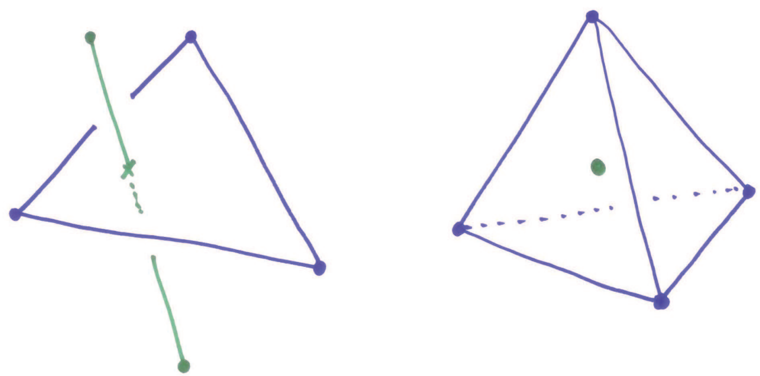
123	~~45~~	56
234	56	~~61~~
345	~~61~~	12
456	12	~~23~~
561	~~23~~	34
612	34	~~45~~

Now let K be an embedded K_6 s.t. the cycle $< 123456 >$ is trefoil. Then (i) can be proved by showing the other Hamiltonian cycles do not satisfy the table.

Rmk. Recently R. Nikkuni reproved Theorem 1 by interpreting the sum of Arf invaraints over all Hamiltonian cycles and the sum of linking numbers of all pairs of two disjoint cycles.

- $\mathcal{K}_n = \{$ all linear embeddings of $K_n\}$
- For $K \in \mathcal{K}_n$, $C(K) =$ number of knots with polygon index n in K
- $M(n) = Max\ \{C(K) | K \in \mathcal{K}_n\}$
- $m(n) = Min\ \{C(K) | K \in \mathcal{K}_n\}$
- By Theorem 1, $M(6) = 1$.
- $m(n) = 0$ for any $n \geq 6$.
- $M(7)$= maximal number of 4_1 knots in linear embeddings of K_7 ??

- Let V be a set of points in $\mathbb{R}^3$.
- Then a partition $V = V_1 \cup V_2$ is a Radon partition, if $\mathsf{Conv}(V_1) \cap \mathsf{Conv}(V_2) \neq \emptyset$
- For example, if $V =$ a set of five points in general position then two kinds of Radon partitions are possible.



- Oriented Matroid Theory
  The relative positions of seven points in general position can be described by an uniform acyclic oriented matroid of rank 4 on seven elements which is in fact a collection of Radon partitions formed by the seven points.
  All these UAOM's are completely listed.

- [Alfonsín] $r(\text{trefoil}) = 7$
  To prove this, he derived several sufficient conditions for a heptagonal knot to be trefoil.
  Each of these condition is described by a set of Radon partitions formed by the vertices of heptagonal trefoil knot.
  And by help of computer, he checked that any UAOM(4,7) satisfies the condition.

- To determine $M(7)$ we can apply this method.

[Theorem 2.] Let P be a heptagonal knot such that its vertices are in general position. Then P is figure-8 if and only if the vertices of P can be labelled so that the polygon satisfies one among three types:

123	45	56	67
	$\pm$	$\mp$	$\times$
234	56	67	71
	$\mp$	$\times$	$\times$
345	67	71	12
	$\times$	$\pm$	$\times$
456	71	12	23
	$\pm$	$\times$	$\times$
567	12	23	34
	$\times$	$\mp$	$\times$
671	23	34	45
	$\mp$	$\times$	$\times$
712	34	45	56
	$\times$	$\pm$	$\times$

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	$\pm$	$\mp$	$\times$
234	56	67	71
	$\mp$	$\times$	$\times$
345	67	71	12
	$\times$	$\pm$	$\times$
456	71	12	23
	$\pm$	$\times$	$\times$
567	12	23	34
	$\times$	$\mp$	$\times$
671	23	34	45
	$\mp$	$\pm$	$\times$
712	34	45	56
	$\times$	$\pm$	$\times$

123	45	56	67
	$\pm$	$\mp$	$\times$
234	56	67	71
	$\times$	$\mp$	$\times$
345	67	71	12
	$\times$	$\pm$	$\times$
456	71	12	23
	$\pm$	$\times$	$\times$
567	12	23	34
	$\times$	$\mp$	$\times$
671	23	34	45
	$\mp$	$\times$	$\times$
712	34	45	56
	$\times$	$\pm$	$\times$

Proof of Theorem 2- Sketch:

Let P be a heptagonal figure-8 knot with vertices $\{1, 2, \ldots, 7\}$ in general position.

Then from the irreducibility of P, we can derive several relations on the Radon partitions formed by the vertices.

These realtions enable us to fill the table completely, which results in one among three types of tables uptp some cyclic permutations of the labels.

123	45	56	67
234	56	67	71
345	67	71	12
456	71	12	23
567	12	23	34
671	23	34	45
712	34	45	56

Lemma 1.

(i)

123	45	56	67
	$\pm$	$\times$	$\times$

$\Longrightarrow$

234	56	67	71
	$\times$		
345	67	71	12
	$\mp$		

(ii)

123	45	56	67
	$\times$	$\times$	$\pm$

$\Longrightarrow$

671	23	34	45
			$\mp$
712	34	45	56
			$\times$

Lemma 2.

(i) $\epsilon(123, 56) = \pm 1$ and $\epsilon(456, 12) \neq 0 \quad \Longrightarrow \quad \epsilon(456, 12) = \pm 1$

(ii) $\epsilon(123, 56) = \pm 1$ and $\epsilon(567, 23) \neq 0 \quad \Longrightarrow \quad \epsilon(567, 23) = \pm 1$

Lemma 3.

(i)

123	45	56	67
	$\bullet$	$\bullet$	

$\Longrightarrow$

345	67	71	12
			$\times$
456	71	12	23
	$\bullet$	$\times$	$\times$

(ii)

123	45	56	67
		$\bullet$	$\bullet$

$\Longrightarrow$

567	12	23	34
	$\times$	$\times$	$\bullet$
671	23	34	45
	$\times$		

Lemma 4. P does not allow any of two cases in the below:

(i)

123	45	56	67
	$\times$	$\pm$	$\times$
345	67	71	12
	$\times$	$\pm$	

(ii)

671	23	34	45
		$\pm$	$\times$
123	45	56	67
	$\times$	$\pm$	$\times$

Lemma 5. There exists no integer i such that $I(i) \geq 2$ and $I(i+1) \geq 2$.

Lemma 6. There exists no pair of distinct integers (i, j) such that

i, i+1, i+2	i+3, i+4	i+4, i+5	i+5, i+6
	$\bullet$	$\bullet$	$\times$
j, j+1, j+2	j+3, j+4	j+4, j+5	j+5, j+6
	$\times$	$\bullet$	$\bullet$

Lemma 7.

(i) For every i, $I(i) \geq 1$.

(ii) There exists an integer i such that $I(i) \geq 2$.

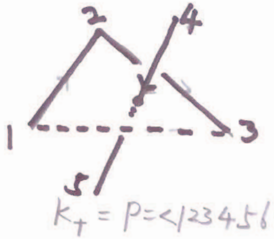
(iii) For every i, $I(i) < 3$.

(iv) If $I(i) = 2$ for some i, then $e_{i+4,i+5}$ should penetrate $\Delta_{i,i+1,i+2}$.

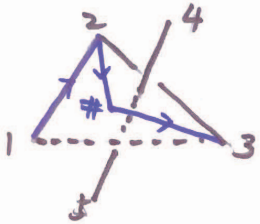
proof of Lemma 1.

Supp

	45	56	67
123	+	×	×

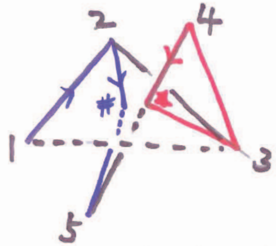


$K_+ = P = \langle 1234567 \rangle$



$K_- = \langle 12\#34567 \rangle$

$\sim \langle 134567 \rangle \to 3_1$ or 0



$K_0 = \langle 12\#567 \rangle \sqcup \langle 34* \rangle$

$\sim \langle 1\#567 \rangle \sqcup \langle 34* \rangle$

$\nabla(K_+) - \nabla(K_-) = t\,\nabla(K_0)$

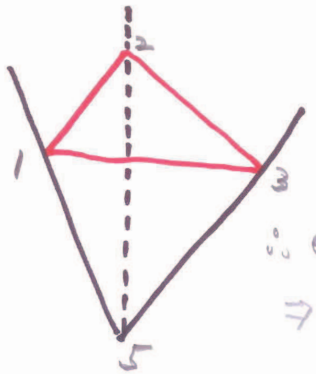
$1-t^2 \qquad 1$ or $1+t^2$

$\Rightarrow \nabla(K_0) = -t$ or $-2t \Rightarrow lk(K_0) = -1$ or -2

$\Delta_{34*} \leftarrow$ ~~56~~, 67, ~~71~~ $\left(\because \in(34*, 71) \neq 0 \Rightarrow \varepsilon(34*, 71) = 1 \right)$

$\left(\because \Delta_{34*} \subset \Delta_{345} \right)$

$\therefore \in(34*, 67) = -1$ & $\underline{\underline{\varepsilon(345, 67) = -1}}$



$T = \{ \overrightarrow{5x} \mid x \in \Delta_{123} \}$

$4 \in T$

$\therefore \in(234, 56) \neq 0$

$\Rightarrow \in(123, 56) \neq \phi$

Contradiction!!