The Heegaard genera of surface sums

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1 Definitions

Compression body. A compression body is a 3-manifold $W$ obtained from a closed connected orientable surface $F$ by attaching 2-handles to $F \times [0, 1]$ along $F \times \{0\}$ and capping off any resulting 2-sphere boundary components with 3-handles.

\[ \partial_+ W = F \times \{1\}. \]

\[ \partial_- W = \partial W - \partial_+ W. \]

Specially, we say $W$ is a handlebody if $\partial_- W = \emptyset$. 
Heegaard splitting. Let $M$ be a 3-manifold. If there is a closed surface $S$ which cuts $M$ into two compression bodies $V$ and $W$, then we say $M$ has a Heegaard splitting $M = V \cup_S W$.

$S$ is called a Heegaard surface of $M$.

If the genus $g(S)$ of $S$ is minimal among all the Heegaard surfaces of $M$, then $g(S)$ is called the genus of $M$, denoted by $g(M)$. 
Surface sum of manifolds. Let $M$ be a compact orientable 3-manifold, and $F$ be a separating incompressible surface in $M$ which cuts $M$ into two 3-manifolds $M_1$ and $M_2$. Then $M$ is called the surface sum of $M_1$ and $M_2$ along $F$, denoted by $M = M_1 \cup_F M_2$.

Self surface sum of manifolds. If $F$ is a non-separating incompressible surface in $M$ which cuts $M$ into a 3-manifold $M_1$, then $M$ is called the self surface sum of $M_1$ along $F$, denoted by $M = M_1 \cup_F$. 
2 The Heegaard genus of surface sums

$F$ is a closed surface:

Let $M = M_1 \cup_F M_2$, and $M_i = V_i \cup S_i W_i$ be a Heegaard splitting for $M_i$ for $i = 1, 2$. Suppose that $F = \partial - W_1 = \partial - V_2$.

$M = \partial V_1 \times I \cup \{1 - \text{handles in } V_1\} \cup \{1 - \text{handles in } V_2\} \cup \{2 - \text{handles in } W_1\} \cup \{2 - \text{handles in } W_2\} \cup \partial - W_2 \times I$.

Then $M$ has a natural Heegaard splitting $M = V \cup W$ called the amalgamation of $V_1 \cup S_1 W_1$ and $V_2 \cup S_2 W_2$.

$g(M) \leq g(M_1) + g(M_2) - g(F)$.
$F$ is a 2-sphere:

$$g(M) = g(M_1) + g(M_2) \text{(Haken’s Lemma)}$$

For $g(F) > 0$:

There are some examples to show that it is possible that

$$g(M) \leq g(M_1) + g(M_2) - g(F) - n$$

for any given $n > 0$. (T. Kobayashi, R. Qiu, Y. Rieck and S. Wang; J. Schultens and R. Weidman)
There are also examples to show that:

\[ g(M) = g(M_1) + g(M_2) - g(F), \]
under various different conditions describing the complicated gluing maps (D. Bachman, S. Schleimer and E. Sedgwick; M. Lackenby; T. Li; and J. Souto).

\[ g(M) = g(M_1) + g(M_2) - g(F) \]
if both \( M_1 \) and \( M_2 \) have high distance Heegaard splittings, where the distance of a Heegaard splitting was introduced by Hempel (T. Kobayashi and R. Qiu).
3 Bounded surface sums

For $F$ is a bounded surface

Let $M = M_1 \cup_F M_2$. Let $M_i = V_i \cup_{S_i} W_i$ be a Heegaard splitting of $M_i$ such that $F \subset \partial_i \subset \partial_- W_i$ and $\partial_i \times [0, 1]$ is disjoint from $S_i$. Now let $r_i$ be an unknotted arc in $W_i$ such that $\partial_1 r_i \subset \partial_+ W_i$, $\partial_2 r_1 = \partial_2 r_2 \subset \text{int} F$. Let $N(r_1 \cup r_2)$ be a regular neighborhood of $r_1 \cup r_2$ in $W_1 \cup_F W_2$. Let $V = V_1 \cup N(r_1 \cup r_2) \cup V_2$, and $W$ be the closure of $(W_1 \cup_F W_2) - N(r_1 \cup r_2)$. Then $M = V \cup W$ is a Heegaard splitting, which we say the surface sum of Heegaard splittings $M_1 = V_1 \cup W_1$ and $M_2 = V_2 \cup W_2$ along $F$. 
Figure 1.

\[ g(M) \leq g(M_1) + g(M_2). \]
For bounded surface $F$:

$F$ is a disk:

$$g(M) = g(M_1) + g(M_2)$$ (disk version of Haken’s lemma)

$F$ is an annulus: $g(M) = g(M_1) + g(M_2)$ or $g(M) < g(M_1) + g(M_2)$. (tunnel number) (T. Kobayashi; T. Kobayashi and Y. Rieck; K. Morimoto; J. Schultens; R. Qiu, K. Du, J. Ma and M. Zhang,...)
4 Reconstruction of the manifolds

Let $M = M_1 \cup_F M_2$, $\partial_i$ be the component of $\partial M_i$ containing $F$, and $\partial_i \times [0, 1]$ be a regular neighborhood of $\partial_i$ in $M_i$ with $\partial_i = \partial_i \times \{0\}$. We denote by $P^i$ the surface $\partial_i \times \{1\}$. Let $M^i = M_i - \partial_i \times [0, 1]$ for $i = 1, 2$, and $M^* = \partial_1 \times [0, 1] \cup_F \partial_2 \times [0, 1]$. Then $M = M^1 \cup_{P^1} M^* \cup_{P^2} M^2$. 
Figure 2.
5 The Heegaard splittings of $M^*$

$M^*$ can be viewed as a surface sum of two I-bundles

$$M^* = \partial_1 \times I \cup_F \partial_2 \times I$$

Figure 3.
Figure 4.

\[ M^* = \partial_1 \times I \cup \partial_3 \times I \]

\[ M^* = \partial_2 \times I \cup \partial_3 \times I \]
Three traditional Heegaard splittings of $M^*$:

$M^* = V_1 \cup_{S_1} W_1.$

$g(S_1) = g(P_1) + g(P_2) = g(\partial_1) + g(\partial_2).$

$M^* = V_2 \cup_{S_2} W_2.$

$g(S_2) = g(P_1) + g(P_3) = 2g(\partial_1) + g(\partial_2) - \chi(F) + 1.$

$M^* = V_3 \cup_{S_3} W_3.$

$g(S_3) = g(P_2) + g(P_3) = g(\partial_1) + 2g(\partial_2) - \chi(F) + 1.$
6 Theorems

Theorem 1. Let $M$ be a surface sum of two irreducible, $\partial$-irreducible 3-manifolds $M_1$ and $M_2$ along a bounded connected surface $F$, and $\partial_i$ be the component of $\partial M_i$ containing $F$. If both $M_1$ and $M_2$ have Heegaard splittings with distance at least $2(g(M_1) + g(M_2)) + 1$, then any minimal Heegaard splitting of $M$ is the amalgamation of Heegaard splittings of $M_1$, $M_2$ and $M^*$ along $P^1$ and $P^2$. (where $M^1$, $M^2$, $P^1$, $P^2$ as just defined)
**Theorem 2.** Under the assumptions of Theorem 1, if $\partial_i - F$ is connected for $i = 1, 2$, then

$$g(M) = \min\{g(M_1) + g(M_2), \alpha\},$$

where $\alpha = g(M_1) + g(M_2) + \frac{1}{2}(2\chi(F) + 2 - \chi(\partial_1) - \chi(\partial_2)) - \max\{g(\partial_1), g(\partial_2)\}$.

Furthermore $g(M) = g(M_1) + g(M_2)$ if and only if $\chi(F) \geq \frac{1}{2} \max\{\chi(\partial_1), \chi(\partial_2)\}$. 
Theorem 3. Let $M$ be a surface sum of two irreducible, $\partial$-irreducible 3-manifolds $M_1$ and $M_2$ along an annulus. If both $M_1$ and $M_2$ have Heegaard splittings with distance at least $2(g(M_1) + g(M_2)) + 1$, then $g(M) = g(M_1) + g(M_2)$. 
Theorem 4. Let $M$ be a self surface sum of an irreducible, $\partial$-irreducible 3-manifold $M_1$ along a compact surface $F$. If $M_1$ has a Heegaard splitting $V_1 \cup_{S_1} W_1$ such that the two copies of $F$ obtained by cutting $M$ along $F$ lie in the same side of $S_1$ and $d(S_1) \geq 2g(M_1) + 2$, then $g(M) = g(M_1) + 1$. 
Thanks!