Regular-equivalence of 2-knot diagrams and sphere eversions

Kokoro TANAKA
Tokyo Gakugei University

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joint work with
Masamichi TAKASE (Seikei University)

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§ Basic definition (2-knots & Diagrams)
a 2-knot/surface-knot
def a submanifold of $\mathbb{R}^4$ which is diffeomorphic to
the 2-sphere $S^2$/an oriented connected closed surface

$F (\subset \mathbb{R}^4)$: a 2-knot/surface-knot, $\pi : \mathbb{R}^4 \to \mathbb{R}^3$: a natural projection
We may assume that $\pi|_F$ is generic.

a diagram of $F$
def the projection image $\pi(F)$ + “height information”

- a regular point
- a double point
- a triple point
- a branch point
Example: A diagram of a 2-knot

branch points

triple points

branch points
Roseman moves

Theorem ([Roseman, 1998])

\[ \begin{align*}
\mathbb{R}^4 & \xrightarrow{\text{proj.}} \mathbb{R}^3 \\
\{ \text{Surface-knots} \} / \text{amb. iso.} & \xrightarrow{1:1} \{ \text{Diagrams} \} / \text{Roseman moves}
\end{align*} \]
$D$: an oriented diagram of a 2-knot/surface-knot

$D$: a **regular** diagram $\xleftrightarrow{\text{def}}$ The diagram $D$ has no branch points.

**Caution**: This example has no triple points, but a regular diagram may have triple points in general.

**Remark**: $\forall$ surface-knot has a regular diagram.

$D_0, D_1$: two oriented (regular) diagrams of a 2-knot/surface-knot

$D_0 \sim \text{r.e.} \ D_1$ (Two diagrams $D_0$ and $D_1$ are **regular-equivalent**.)

$\xleftrightarrow{\text{def}}$ $\exists$ sequence of “branch-free” Roseman moves between $D_0$ and $D_1$
§ Prelude (1-knot case)
Question (for 1-knot diagrams)

Question

$D_0, D_1(\subset \mathbb{R}^2)$: oriented diagrams of a 1-knot

$w(D_0) = w(D_1) \quad \Rightarrow \quad D_0 \approx D_1$

$w(\cdot)$: the writhe,

$r.e.$: regular-equivalent $= \exists$ sequence of R2 and R3
**Question (for 1-knot diagrams)**

**Question**

\[ D_0, D_1(\subset \mathbb{R}^2): \text{oriented diagrams of a 1-knot} \]

\[ w(D_0) = w(D_1) \overset{??}{\implies} D_0 \overset{\text{r.e.}}{\sim} D_1 \]

\[ w(\cdot): \text{the writhe,} \]

\[ \overset{\text{r.e.}}{\sim}: \text{regular-equivalent} = \exists \text{ sequence of R2 and R3} \]

**Answer: NO!! (well-known)**

\[ D_0: \begin{array}{c} \includegraphics[width=0.3\textwidth]{diagram1.png} \end{array}, \]

\[ D_1: \begin{array}{c} \includegraphics[width=0.3\textwidth]{diagram2.png} \end{array} \implies D_0 \overset{\text{r.e.}}{\not\sim} D_1. \]

\[ \quad \text{diagrams of a (trivial) 1-knot} \]
Proof

Theorem ([Whitney, 1937], [Graustein])

\( P_0, P_1 (\subset \mathbb{R}^2) \): oriented immersed curves

\[ P_0 \overset{r.h.}{\sim} P_1 \iff r(P_0) = r(P_1) \]

\( r.h. \): regularly homotopic, \( r(\cdot) \): the rotation number (\( \scriptstyle{\circlearrowleft} = +1 \), \( \scriptstyle{\circlearrowright} = -1 \))
Proof

**Theorem ([Whitney, 1937], [Graustein])**

\( P_0, P_1 (\subset \mathbb{R}^2) \): oriented immersed curves

\[ P_0 \stackrel{r.h.}{\sim} P_1 \iff r(P_0) = r(P_1) \]

\( r.h. \): regularly homotopic, \( r(\cdot) \): the rotation number (\( \leftarrow = +1, \rightarrow = -1 \))

\[ D_0: \quad \xrightarrow{\text{Forget over/under info.}} \quad D_1: \]

Assume \( D_0 \stackrel{r.e.}{\sim} D_1 \).

\[ D_0 \stackrel{r.h.}{\sim} D_1 \]

\( D(\subset \mathbb{R}^2) \): a diagram \( \sim D \): the underlying immersed curve of \( D \)
$D_0, D_1 (\subset \mathbb{R}^2)$: oriented diagrams of a 1-knot

\[ w(D_0) = w(D_1) \quad \implies \quad D_0 \sim D_1 \]

$w(\cdot)$: the writhe,

$\sim$: regular-equivalent = \exists$ sequence of R2 and R3
**Question**

\( D_0, D_1 (\subset \mathbb{R}^2) \): oriented diagrams of a 1-knot

\[ w(D_0) = w(D_1) \implies D_0 \overset{r.e.}{\sim} D_1 \]

\( w(\cdot) \): the writhe,

\( r.e. \): regular-equivalent = \( \exists \) sequence of R2 and R3

**Theorem ([Trace, 1983])**

\( D_0, D_1 (\subset \mathbb{R}^2) \): oriented diagrams of a 1-knot

\[ D_0 \overset{r.e.}{\sim} D_1 \iff w(D_0) = w(D_1) \text{ and } r(D_0) = r(D_1). \]

\( D(\subset \mathbb{R}^2) \): a diagram \( \sim D \): the underlying immersed curve of \( D \)

\( r(\cdot) \): the rotation number \( (\bowtie = +1, \bowtie = -1) \)
\S Problem (2-knot case)
Question (for 2-knot diagrams)

**Question**

\[ D_0, D_1 (\subset \mathbb{R}^3) : \text{oriented diagrams of a 2-knot} \]

\[ b(D_0) = b(D_1) \implies D_0 \overset{\text{r.e.}}{\sim} D_1 \]

\( b(\cdot) \): \# of branch points,

\( \overset{\text{r.e.}}{\sim} \): regular-equivalent = \exists sequence of “branch-free” Roseman moves

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Regular-equivalence & Sphere eversion  
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Observation (for regular 2-knot diagrams)

Question

\[ D_0, D_1 (\subset \mathbb{R}^3) : \text{oriented diagrams of a 2-knot} \]
\[ b(D_0) = b(D_1) \iff D_0 \overset{\text{r.e.}}{\sim} D_1 \]

Observation

\[ D_0, D_1 (\subset \mathbb{R}^3) : \text{oriented} \quad \widetilde{\quad} \text{regular diagrams of a 2-knot} \]
\[ \iff b(D_0) = b(D_1) = 0 \]

\[ D_0, D_1 \quad \text{immersed} \quad S^2 \quad \text{in} \quad \mathbb{R}^3 \]
\[ D_0 \overset{\text{r.e.}}{\sim} D_1 \text{ implies } D_0 \overset{\text{r.h.}}{\sim} D_1. \]
\[ \iff \text{We may use immersion theory.} \]

Note: \forall 2-knot has a regular diagram. [Carter–Saito]

\[ \iff \text{We consider the above problem for regular diagrams.} \]
Concrete example (of Question)

\[ D_0: \quad \text{regular diagrams of a (trivial) 2-knot} \]

\[ \quad \rightarrow \quad D_0 \sim D_1. \]

It seems that the answer is NO (i.e. \( D_0 \not\sim D_1 \)).

However...
Concrete example (of Question)

\[
\begin{array}{c}
D_0: \\
\quad \\
D_1:
\end{array}
\quad \xrightarrow{??} \quad D_0 \sim \text{r.e.} D_1.
\]

\text{regular diagrams of a (trivial) 2-knot}

It seems that the answer is NO (i.e. \(D_0 \not\approx \text{r.e.} D_1\)).

However...

Assume \(D_0 \overset{\text{r.e.}}{\sim} D_1\).

\[\downarrow \text{Forget over/under info.}\]

\(D_0 \overset{\text{r.h.}}{\sim} D_1\).

Smale’s theorem

\[\downarrow\]

\(D_0 \overset{\text{r.h.}}{\sim} D_1\).

\(\Rightarrow \leftarrow (??)\)

Theorem ([Smale, 1957])

\(\forall\) two immersions of \(S^2\) in \(\mathbb{R}^3\) are regularly homotopic.
§ Related work ($T^2$-knot case)
Regular-equivalence of $T^2$-knot diagrams

**Theorem ([Satoh, 2001])**

\[ D_0: \quad D_1: \quad \implies D_0 \not\sim D_1. \]

Note: \( D_0 \sim D_1 \)

\[ \text{regular diagrams of a (trivial) } T^2\text{-knot} \]
Regular-equivalence of $T^2$-knot diagrams

Theorem ([Satoh, 2001])

$D_0$: \includegraphics[width=2cm]{diagram1.png}, $D_1$: \includegraphics[width=2cm]{diagram2.png} \implies D_0 \not\sim D_1.

Note: $D_0 \sim_{r.h.} D_1$ regular diagrams of a (trivial) $T^2$-knot

How to prove?

Method 1 ([Satoh], Use double point curves of regular diagrams)

$\sim \in H_1(T^2)$: invariant under "branch-free" Roseman moves

Method 2 (Use rack theory)

$\sim \in \mathbb{N} \cup \{0\}$: invariant under "branch-free" Roseman moves

However...
Theorem ([Satoh, 2001])

\[ D_0: \quad D_1: \]

\[ \Rightarrow D_0 \not\approx_r D_1. \]

\textbf{Note:} \( D_0 \) \( \approx_r \) \( D_1 \)

\textit{regular diagrams of a (trivial) } \( T^2 \)-knot

**How to prove?**

\begin{itemize}
  \item \textbf{Method 1} ([Satoh], Use double point curves of regular diagrams)
    \[ \Rightarrow \bullet \in H_1(T^2): \text{invariant under "branch-free" Roseman moves} \]
  \item \textbf{Method 2} (Use rack theory)
    \[ \Rightarrow \bullet \in \mathbb{N} \cup \{0\}: \text{invariant under "branch-free" Roseman moves} \]
\end{itemize}

However...

\textbf{Method 1} doesn’t work well for 2-knots. \( (\because H_1(S^2) = 0. \) \)
\textbf{Method 2} also doesn’t work well for 2-knots. \( [\text{Oshiro–T}] \)
.§ Main theorem & Proof (Rough sketch)
Main theorem (of today)

Main Theorem [Takase–T]

\[ D_0 : \longrightarrow D_1 : \quad \implies \quad D_0 \not\sim^r D_1. \]

\[ \text{regular diagrams of a (trivial) 2-knot} \]

By Smale's theorem, we have

\[ D_0 \text{ r.h. } D_1 \text{ (sphere eversion).} \]

Corollary

No sphere eversion can be lifted to an isotopy in \( \mathbb{R}^4 \).
Main theorem (of today)

Main Thorem [Takase–T]

\[ D_0 : \quad D_1 : \quad \implies D_0 \text{ r.e. } D_1. \]

\[ \blacktriangleleft \text{regular diagrams of a (trivial) 2-knot} \]

By Smale’s theorem, we have \( D_0 \overset{\text{r.h.}}{\sim} D_1 \) (sphere eversion).

Corollary

No sphere eversion can be lifted to an isotopy in \( \mathbb{R}^4 \).
Sketch of Proof

Assume $D_0 \sim D_1$.

There exists an isotopy $\tilde{h}_t : S^2 \to \mathbb{R}^4$ such that $h_t = \pi \circ \tilde{h}_t : S^2 \to \mathbb{R}^3$ is a regular homotopy between $h_0(S^2) = D_0$ and $h_1(S^2) = D_1$.

Claim (two contradictory claims)

1. The number of quadruple points of the track of $\{h_t\}$ is $1 \pmod{2}$
2. The number of quadruple points of the track of $\{h_t\}$ is $0 \pmod{2}$
Quadruple points of the track of \( \{h_t\} \)

Track of T2 \( \iff \) Quadruple point in \( D^3 \times [0, 1] \)
(\( \nearrow \) without over/under information)

1-knot case:
Track of R3 \( \iff \) Triple point in \( D^2 \times [0, 1] \)
(\( \nearrow \) without over/under information)
Proof of Claim 1

\[ D_0: \quad \text{\raisebox{-0.5cm}{\includegraphics[width=0.2\textwidth]{sphere1.png}}} \quad , \quad D_1: \quad \text{\raisebox{-0.5cm}{\includegraphics[width=0.2\textwidth]{sphere2.png}}} \quad \implies \quad D_0 \not\sim r.e. D_1. \]

\[ \exists \text{ isotopy } \tilde{h}_t: S^2 \hookrightarrow \mathbb{R}^4 \quad \text{s.t.} \quad h_t = \pi \circ \tilde{h}_t: S^2 \hookrightarrow \mathbb{R}^3 \text{ is a regular homotopy} \]

between \( h_0(S^2) = D_0 \) and \( h_1(S^2) = D_1. \)

Claim 1: \# of quadruple points of the track of \( \{h_t\} \equiv 1 \mod 2 \)

The regular homotopy \( \{h_t\} \) is a sphere eversion.

Then Claim 1 follows from the following.

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**Theorem ([Max–Banchoff, 1980], [Hughes, 1985])**

\[ \forall \text{ sphere eversion has odd } \# \text{ of quadruple points.} \]
Proof of Claim

Claim 2: \# of quadruple points of the track of \( \{h_t\} \equiv 0 \pmod{2} \)

\[
\begin{align*}
\mathbb{R}^4 \times [0, 1] & \quad \mathbb{R}^3 \times [0, 1] \\
\text{track of } \tilde{h}_t & \quad \text{track of } h_t
\end{align*}
\]

\[
\begin{align*}
\tilde{F} : S^3 & \xrightarrow{\text{emb.}} \mathbb{R}^5 \\
F : S^3 & \xrightarrow{\text{imm.}} \mathbb{R}^4
\end{align*}
\]

\( F = \pi \circ \tilde{F} \leadsto \) We can show that \( F \) is a null-cobordant immersion.

[Frederman, 1978]

\[
\{M^3 \hookrightarrow \mathbb{R}^4\}/\text{cob.} \left( \cong \mathbb{Z}/24\mathbb{Z} \right) \xrightarrow{\# \text{ of quad. pts}} \mathbb{Z}/2\mathbb{Z} \pmod{2}
\]
§ Generalization
Theorem 1

**Theorem 1 [Takase–T]**

1. $D(\subseteq \mathbb{R}^3)$: an oriented regular diagram of a (+)-amphchiral 2-knot
   $\implies D \not\sim D^* (D^*$: the **mirrored** diagram of $D$)

2. $D(\subseteq \mathbb{R}^3)$: an oriented regular diagram of an **invertible** 2-knot
   $\implies D \not\sim -D (-D$: the **inverted** diagram of $D$)

---

Main theorem is a special case of Theorem 1.

**Main Thorem [Takase–T]**

$D_0$: regular diagrams of a (trivial) 2-knot

$D_1$: regular diagrams of a (trivial) 2-knot

$\implies D_0 \not\sim D_1$
**Theorem 2 [Takase–T]**

\[ D(\subset \mathbb{R}^3) \]: an oriented regular diagram of a 2-knot

\[ D_\infty \]: an **everted** diagram of \( D \)

\[ \implies D \overset{r.e.}{\not\sim} D_\infty \] (Note: \( D \) and \( D_\infty \) represent the same 2-knot.)

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\[ D_0(\subset \mathbb{R}^3) \]: a regular diagram of the trivial 2-knot obtained by spinning the tangle diagram

\( D_\infty \) is well-defined up to regular-equivalence.
Thank you for your attention!