On Heegaard splittings of knot exteriors with tunnel number degenerations

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$K$ : a knot in $S^3$
$E(K)$ : the exterior
$\gamma_1, \gamma_2, \cdots, \gamma_t$ : arc system properly embedded in $E(K)$

The arc system is called an unknottting tunnel system (u. t. s) if the complementary space is homeomorphic to a handlebody.

**Definition**

$t(K) = \text{the minimal number among all u. t. s. of } K$.
We call $t(K)$ the tunnel number of $K$.

By the definition, we see that the u. t. s. corresponds to a Heegaard splitting of $E(K)$, i.e., $t(K) = g(E(K)) - 1$. 
Example
The following knot is $8_{16}$, and the tunnel number is 2. The system of two red arcs is an u. t. s.
$K_1, K_2$: two knots in $S^3$

$K_1\#K_2$: the connected sum of $K_1, K_2$

There had been the following problem on the additivity of tunnel numbers.

**Problem**

Are there knots whose tunnel numbers go down under connected sum?.

i. e.

$t(K_1\#K_2) < t(K_1) + t(K_2)$?
The first result was:

**Theorem 1 (’92 M.)**

Let $K_n$ be the knot as in the figure ($n \neq 0$).

Then $t(K_n) = 2$ and $t(K_n \# K) = 2$ for any 2-bridge knot $K$.

Thus we have: $t(K_n \# K) = 2 < 3 = t(K_n) + t(K)$. 

![Diagram of $K_n$ and $2n + 1$]
By the above result,
We can ask what kind of types such knots are.
On this question,
we showed the following characterization theorem:

**Theorem 2 (’94 M.)**
Suppose \( t(K_1) = 1, t(K_2) = 2 \) and \( t(K_1 \# K_2) = 2 \). Then,
\( K_1 \) is a 2-bridge knot and
\( K_2 \) is a knot with a 2-string essential free tangle decomposition such that at least one of the two tangles has an unknotted component.
Examples

2-string essential free tangles.

with an unknotted component

with no unknotted component
Examples

Knots with a 2-string essential free tangle decomposition.
In the present talk, we consider genus three Heegaard splittings of the knot exterior $E(K_1 \# K_2)$ for the two knots in Theorem 2, and classify the unknotted tunnel systems up to homeomorphisms.

First we get:

**Theorem 3**

Let $K_1$ and $K_2$ be the two knots in Theorem 2.

Then any genus three Heegaard splitting of $E(K_1 \# K_2)$ is strongly irreducible.
Next, concerning the homeomorphism classes of those Heegaard splittings, we get:

**Theorem 4**

Let $K_1$ and $K_2$ be the two knots in Theorem 2.

Then $E(K_1 \# K_2)$ contains at most four genus three Heegaard splittings up to homeomorphism.
To give the complete classification of those four genus three Heegaard splittings, we assume:

\( K_1 \) is a 2-bridge knot \( S(\alpha, \beta) \) (Schubert’s notation).
\( K_2 \) has a 2-string essential free tangle decomposition

\[
(S^3, K_2) = (C_1, t_1 \cup s_1) \cup (C_2, t_2 \cup s_2)
\]
such that

\( C_1 \) contains an unknotted component, i.e., \( t_1 \) or \( s_1 \).
To state the classification theorem, we put the following cases:

Case 1: $C_2$ contains no unknotted component.
Case 2: $C_2$ contains an unknotted component, i.e., $t_2$ or $s_2$.

In Case 2, we have the following two subcases:

Case 2a: there is a homeomorphism exchanging the two tangles $(C_1, t_1 \cup s_1)$ and $(C_2, t_2 \cup s_2)$.
Case 2b: there is no homeomorphism exchanging the two tangles $(C_1, t_1 \cup s_1)$ and $(C_2, t_2 \cup s_2)$.

Then we get:
Theorem 5
Let $K_1$ and $K_2$ be the two knots in Theorem 2. Then we have the following complete classification of genus three Heegaard splittings of $E(K_1\#K_2)$ up to homeomorphism, where $n$ is the number of homeomorphism classes.

Case 1
\[
\begin{align*}
  n &= 1 \quad \text{if } \beta \equiv \pm 1 \pmod{\alpha} \\
  n &= 2 \quad \text{if } \beta \not\equiv \pm 1 \pmod{\alpha}
\end{align*}
\]

Case 2a
\[
\begin{align*}
  n &= 1 \quad \text{if } \beta \equiv \pm 1 \pmod{\alpha} \\
  n &= 2 \quad \text{if } \beta \not\equiv \pm 1 \pmod{\alpha}
\end{align*}
\]

Case 2b
\[
\begin{align*}
  n &= 2 \quad \text{if } \beta \equiv \pm 1 \pmod{\alpha} \\
  n &= 4 \quad \text{if } \beta \not\equiv \pm 1 \pmod{\alpha}
\end{align*}
\]
Examples
In the following, we show some examples of the unknotting tunnel systems of $K_1 \# K_2$ corresponding to the Heegaard splittings of the following cases in the table of Theorem 5.

Case 1  $n = 2$ if $\beta \not\equiv \pm 1 \pmod{\alpha}$
Case 2b $n = 4$ if $\beta \not\equiv \pm 1 \pmod{\alpha}$
$K_1$: a 2-bridge knot $S(23, 7)$ i.e., $\beta \not\equiv \pm 1 \pmod{\alpha}$
$K_2$: a knot with a free tangle decomposition in Case 1.
$K_3$: a knot with a free tangle decomposition in Case 2b.
Case 1

\[ n = 2 \]

\[ K_1 \# K_2 \]
Case 2b
\[ n = 4 \]
\[ K_1 \# K_3 \]
Animation

Unknotting tunnel

system of $K_1 \# K_2$