Bipartite Intrinsically Knotted Graphs with 22 Edges

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Knots and Low Dimensional Manifolds
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Introductions

We will take an embedded graph to mean a graph embedded in $\mathbb{R}^3$.

- A graph $G$ is *intrinsically knotted* (IK) if every embedding of the graph contains a non-trivially knotted cycle.

- [Conway and Gordon]
  Every embedding of $K_7$ contains a knotted cycle.

- [Foisy]
  $K_{3,3,1,1}$ is an intrinsically knotted graph.
Introductions

- A graph $H$ is *minor* of another graph $G$ if $H$ can be obtained from $G$ by edge contracting or edge deleting some edges.

- A graph $G$ is intrinsically knotted and has no proper minor which is intrinsically knotted, $G$ is said to be *minor minimal intrinsically knotted*.

- [Robertson and Seymour]
  There are only finite number of minor minimal intrinsically knotted graphs.

Open Problem

Finding the complete set of minor minimal intrinsically knotted graphs.
- **∇-Y move** and **Y-∇ move**

- [Motwani, Raghunathan, and Saran]
  \( \nabla - Y \) move preserves intrinsic knottedness.

- [Flapan and Naimi]
  Some \( Y - \nabla \) moves do not preserve intrinsic knottedness.

- If \( G' \) is obtained from \( G \) by some \( \nabla - Y \) or \( Y - \nabla \) moves then \( G \) and \( G' \) are **cousin**. The set of all cousins of \( G \) is called the **\( G \) family**.
$E_9 + e$
$E_9 + e$

$K_7 \rightarrow H_8 \rightarrow H_9 \rightarrow H_{10} \rightarrow H_{11} \rightarrow H_{12}$

$F_9 \rightarrow F_{10} \rightarrow F_{10}$

$E_{10} \rightarrow E_{11}$

$C_{11} \rightarrow C_{12} \rightarrow C_{13} \rightarrow C_{14}$

$N_9 \rightarrow N_{10} \rightarrow N_{11} \rightarrow N_{12}$

$N'_9 \rightarrow N'_{10} \rightarrow N'_{11} \rightarrow N'_{12}$
Main Theorem

- A bipartite graph is a graph whose vertices can be divided into two disjoint sets $A$ and $B$ such that every edge connects a vertex in $A$ to one in $B$.

- [Lee, Kim, Lee, and Oh], [Barsotti and Mattman]
  The only triangle-free intrinsically knotted graphs with 21 edges are $H_{12}$ and $C_{14}$.

Main Theorem

There are exactly two bipartite intrinsically knotted graphs with 22 edges.
Two bipartite intrinsically knotted graphs with 22 edges

Figure: Cousin 89 of the $E_9 + e$ family

Figure: Cousin 110 of the $E_9 + e$ family
Terminology

For any two distinct vertices $a$ and $b$,

- $G = (A, B, E)$: a bipartite graph with 22 edges whose partition has the parts $A$ and $B$.

- $\hat{G}_{a,b} = (\hat{V}_{a,b}, \hat{E}_{a,b})$: the graph obtained from $G\{a, b\}$ by contracting one edge incident to a vertex of degree 1 or 2 repeatedly until no vertices of degree 1 or 2 exist. (removing one vertex means deleting interiors of all edges incident to it.)

- $\text{dist}(a, b)$: the distance between two vertices $a$ and $b$ says the number of edges in the shortest path connecting them.

- $\text{deg}(a)$: the degree of a vertex $a$. 
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- $E(a)$ is the set of edges which are incident to $a$. 

$$E(a)$$
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- $V(a) = \{ c \in V \mid \text{dist}(a, c) = 1 \}$
Terminology

To count the number of edges of \( \hat{G}_{a,b} \), we have some notations.

- \( V_n(a) = \{ c \in V | \text{dist}(a, c) = 1, \text{deg}(c) = n \} \)
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- $V_n(a, b) = V_n(a) \cap V_n(b)$.
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- $V_Y(a,b) = \{ c \in V \mid \exists d \in V_3(a,b) \text{ such that } c \in V_3(d) \setminus \{a, b\} \}$
We have a count equation in $\hat{G}_{a,b}$:

$$|\hat{E}_{a,b}| = 22 - |E(a) \cup E(b)| - \left( |V_3(a)| + |V_3(b)| - |V_3(a, b)| + |V_4(a, b)| + |V_Y(a, b)| \right)$$
**Terminology**

A graph is *n-apex* if one can remove *n* vertices from it to obtain a planar graph.

**Lemma 1**

If $G$ is a 2-apex, then $G$ is not intrinsically knotted.

**Lemma 2**

If $|\hat{E}_{a,b}| \leq 8$, then $\hat{G}_{a,b}$ is a planar graph. Thus $G$ is not intrinsically knotted.

**Lemma 3**

If $|\hat{E}_{a,b}| = 9$ and it is not homeomorphic to $K_3, 3$, then $G$ is not intrinsically knotted.
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A graph is \textit{n-apex} if one can remove \(n\) vertices from it to obtain a planar graph.

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- Construction all possible such bipartite graph $G$ with 22 edges,
- Deleting two suitable vertices $a$ and $b$ of $G$,
- Counting the number of edges of $\hat{G}_{a,b}$.

If $|\hat{E}_{a,b}| \leq 9$, we will show that $\hat{G}_{a,b}$ is planar.
If not, we will show that $G$ is an intrinsically knotted graph.

Let $a$ be one of vertices with maximal degree in $G$.
The proof is divided into three parts according to the degree of $a$.

- Any graph $G$ with $\deg(a) \geq 6$ cannot be intrinsically knotted.
- The only bipartite intrinsically knotted graph with $\deg(a) = 5$ is cousin 110 of the $E_9 + e$ family.
- The only bipartite intrinsically knotted graph with $\deg(a) = 4$ is cousin 89 of the $E_9 + e$ family.
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The example of an intrinsically knotted graph.

Cousin 110 of the $E_9 + e$ family
deg(a) = 4

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Thanks for listening