Non-surjective satellite operators and piecewise-linear concordance

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Which knots $K \subset \mathbb{R}^3$ (or $S^3$) can occur as cross-sections of embedded spheres in $\mathbb{R}^4$ (or $S^4$)?
Concordance

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Equivalently, which knots in $\mathbb{R}^3$ (or $S^3$) bound properly embedded disks in $\mathbb{R}^4_+$ (or $D^4$)?
Concordance

Definition

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Knots $K_1, K_2$ are **smoothly/topologically concordant** if they cobound an embedded annulus in $S^3 \times I$, or equivalently if $K_1 \# - K_2$ is topologically/smoothly slice, where $-K = \overline{K}^r$. 
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$C = \{\text{knots}\}/\text{smooth conc.} \quad C^{\text{top}} = \{\text{knots}\}/\text{top. conc.}$
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Conjecture (Zeeman, 1963)

In an arbitrary compact, contractible 4-manifold $X$ other than the 4-ball, not every knot $K \subset \partial X$ bounds a PL disk.
Theorem (Matsumoto–Venema, 1979)

There exists a non-compact, contractible 4-manifold with boundary $S^1 \times \mathbb{R}^2$ such that $S^1 \times \{\text{pt}\}$ does not bound an embedded PL disk.
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Theorem (Akbulut, 1990)

There exist a compact, contractible 4-manifold $X$ and a knot $\gamma \subset \partial X$ that does not bound an embedded PL disk in $X$. 
Akbulut’s example

Akbulut’s manifold $X$ is the original Mazur manifold:

$$X = S^1 \times D^3 \cup_Q 2\text{-}handle,$$

$$Q \subset S^1 \times D^2 \subset \partial(S^1 \times D^3),$$

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- But $\gamma$ bounds a smoothly embedded disk in a different contractible 4-manifold $X'$ with $\partial X' = \partial X$. 

\[ \begin{array}{c}
X' \\
\gamma
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Theorem (L., 2014)

There exist a contractible 4-manifold $X$ and a knot $\gamma \subset \partial X$ such that $\gamma$ does not bound an embedded PL disk in any contractible manifold $X'$ with $\partial X' = \partial X$. 
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In place of the trefoil, can use any knot $J$ with $\epsilon(J) = 1$, where $\epsilon$ is Hom’s concordance invariant.
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  $$\Delta_K(t) = f(t)f(t^{-1})$$ 
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- **Tristram–Levine signatures**
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- **Casson–Gordon invariants**

Freedman: If $\Delta_K(t) \equiv 1$, then $K$ is topologically slice; e.g., Whitehead doubles. But many such knots are not smoothly slice.
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Smooth concordance obstructions

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  - $|\tau(K)| \leq g_4(K)$.
  - Whitehead doubles: If $\tau(K) > 0$, then $\tau(Wh_+(K)) = 1$, so $Wh_+(K)$ is not smoothly slice.

- $\epsilon(K) \in \{-1, 0, 1\}$ (Hom):
  - Sign-additive under connected sum.
  - Vanishes for slice knots.
  - $\mathcal{C}/\ker(\epsilon)$ contains a $\mathbb{Z}\infty$ summand of topologically slice knots.
Expanded notions of smooth concordance

Every knot $K \subset S^3$ bounds a smooth disk in some 4-manifold $X$ with $\partial X = S^3$; for instance, can take $X = (k \mathbb{C}P^2 \# l \overline{\mathbb{C}P^2}) \setminus B^4$. 
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**Definition**

For a ring $R$, $K$ is $R$–homology slice if it bounds a smoothly embedded disk in a smooth 4-manifold $X$ with $\partial X = S^3$ and $\tilde{H}_\ast(X; R) = 0$. 
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- $K$ is **pseudo-slice** or **exotically slice** if it bounds a smoothly embedded disk in a smooth, contractible 4-manifold $X$ with $\partial X = S^3$. (Freedman: $X$ is homeomorphic to $D^4$, but with a potentially exotic smooth structure.)
Let $C_R$ and $C_{ex}$ denote the corresponding concordance groups, so that

$$C \rightarrow C_{ex} \rightarrow C_{\mathbb{Z}} \rightarrow C_{\mathbb{Q}}.$$
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Rasmussen’s invariant $s(K)$ (coming from Khovanov homology) was originally only proven to obstruct honest smooth concordance, but Kronheimer and Mrowka showed it actually descends to $C_{\text{ex}}$. 
Knots $K_1, K_2$ in homology spheres $Y_1, Y_2$ are

**\textit{R–homology concordant}** if there is a smooth $R$-homology cobordism $W$ from $Y_1$ to $Y_2$ (i.e. $H_*(Y_i; R) \xrightarrow{\cong} H_*(W; R)$) and a smooth annulus in $W$ connecting $K_1$ and $K_2$;
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A knot $K \subset Y$ bounds a PL disk in a contractible 4-manifold $X$ iff it is exotically cobordant to a knot in $S^3$, since we can delete a ball containing all the singularities.
**Definition**

Given a pattern knot $P \subset S^1 \times D^2$ and a companion knot $K \subset S^3$, the satellite knot $P(K) \subset S^3$ is the image of $P$ under the Seifert framing $S^1 \times D^2 \hookrightarrow S^3$ of $K$. 

![Diagram of satellite knots](image)
If $K_1$ is concordant to $K_2$, then $P(K_1)$ is concordant to $P(K_2)$; this gives us maps

$$
\begin{array}{cccc}
C & \rightarrow & C^{\text{ex}} & \rightarrow & C^\mathbb{Z} & \rightarrow & C^\mathbb{Q} \\
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Satellite operators are generally not group homomorphisms.
Definition

$P \subset S^1 \times D^2$ has winding number $n$ if it represents $n$ times a generator of $H_1(S^1 \times D^2)$. 

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Non-surjective satellite operators and PL concordance
Definition

- $P \subset S^1 \times D^2$ has winding number $n$ if it represents $n$ times a generator of $H_1(S^1 \times D^2)$.

- $P$ has strong winding number 1 if the meridian $[\{\text{pt}\} \times \partial D^2]$ normally generates $\pi_1(S^1 \times D^2 \setminus P)$. 
Theorem (L., 2014)

There exists a (strong) winding number 1 pattern $P \subset S^1 \times D^2$ such that $P(K)$ is not $\mathbb{Z}$–homology slice for any knot $K \subset S^3$ (including the unknot).
Non-surjective satellite operators

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- It suffices to find a pattern $Q$ such that $Q : C^\mathbb{Z} \to C^\mathbb{Z}$ is not surjective, and set $P = Q \# -J$ for $J \notin \text{im}(Q)$. 
Proof of the main theorem

Let \( P \) be a winding number 1 pattern such that \( P(K) \) is not \( \mathbb{Z} \)-homology slice for any \( K \).
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Let $Y$ be the boundary of the Mazur-type manifold obtained from $P$, and let $\gamma$ be the knot $S^1 \times \{\text{pt}\}$.
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Let $Y$ be the boundary of the Mazur-type manifold obtained from $P$, and let $\gamma$ be the knot $S^1 \times \{\text{pt}\}$.

Suppose $\gamma$ bounds a PL disk $\Delta$ in a contractible 4-manifold $X$ with $\partial X = Y$. Can assume that $\Delta$ has singularities that are cones on knots $K_1, \ldots, K_n \subset S^3$. 
Drill out arcs to see that $\gamma \not\# K$ bounds a smooth slice disk $\Delta' \subset X$, where $K = -(K_1 \not\# \cdots \not\# K_n)$. 
Proof of the main theorem

- Drill out arcs to see that $\gamma \not\# K$ bounds a smooth slice disk $\Delta' \subset X$, where $K = -(K_1 \not\# \cdots \not\# K_n)$.
- Attach a 0-framed 2-handle along $\gamma \not\# K$ to obtain $W$, a homology $S^2 \times D^2$, whose $H_2$ is generated by an embedded sphere $S$ with trivial normal bundle.
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- Surger out $S$ to obtain $W'$, a homology $D^3 \times S^1$.
Proof of the main theorem

- Drill out arcs to see that $\gamma \not\approx K$ bounds a smooth slice disk $\Delta' \subset X$, where $K = -(K_1 \not\approx \cdots \not\approx K_n)$.

- Attach a 0-framed 2-handle along $\gamma \not\approx K$ to obtain $W$, a homology $S^2 \times D^2$, whose $H_2$ is generated by an embedded sphere $S$ with trivial normal bundle.

- Surge out $S$ to obtain $W'$, a homology $D^3 \times S^1$.

- Now $\partial W = \partial W' \cong S^3_0(P(K))$, and $H_1(W')$ is generated by $\lambda$. 
Proof of the main theorem

Attach a 0-framed 2-handle along $\lambda$ to obtain $Z$, a homology $D^4$. The belt circle $\mu$ of this 2-handle bounds a smoothly embedded disk (the cocore).
Proof of the main theorem

- Attach a 0-framed 2-handle along $\lambda$ to obtain $Z$, a homology $D^4$. The belt circle $\mu$ of this 2-handle bounds a smoothly embedded disk (the cocore).
- The boundary of $Z$ is $S^3$, and $\mu = P(K)$. Contradiction!
Let $Q$ denote the Mazur pattern:
Proposition

For any knot $K \subset S^3$,

$$\tau(Q(K)) = \begin{cases} 
\tau(K) & \text{if } \tau(K) \leq 0 \text{ and } \epsilon(K) \in \{0, 1\} \\
\tau(K) + 1 & \text{if } \tau(K) > 0 \text{ or } \epsilon(K) = -1
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Non-surjective satellite operators

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Thus, if $J$ is a knot with $\epsilon(J) = -1$ (e.g. the left-handed trefoil), then $J$ is not homology concordant to $Q(K)$ for any $K$. 

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Thus, if $J$ is a knot with $\epsilon(J) = -1$ (e.g. the left-handed trefoil), then $J$ is not homology concordant to $Q(K)$ for any $K$.

- Proof uses bordered Floer homology, with computations assisted by Bohua Zhan’s Python implementation of Lipshitz, Ozsváth, Thurston’s arc slides algorithm.
Corollary

For any knot $K$ and any $m > 1$, $\tau(Q^m(K)) \not\in \{1, \ldots, m - 1\}$. 
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$$C \supsetneq \text{im}(Q) \supsetneq \text{im}(Q^2) \supsetneq \cdots$$
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- $Q$ has strong winding number 1, so by a theorem of Cochran, Davis, and Ray,

$$Q: C^{ex} \to C^{ex}$$

is injective.
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For any knot $K$ and any $m > 1$, $\tau(Q^m(K)) \notin \{1, \ldots, m-1\}$. Therefore, the action of the Mazur satellite operator $Q$ on $C$, $C^\text{ex}$, or $C^\mathbb{Z}$ satisfies

$$C \supsetneq \text{im}(Q) \supsetneq \text{im}(Q^2) \supsetneq \cdots$$

- $Q$ has strong winding number 1, so by a theorem of Cochran, Davis, and Ray,

$$Q : C^\text{ex} \to C^\text{ex}$$

is injective.

- Hence, the iterates of $Q$ are decreasing self-similarities of $C^\text{ex}$. 